REACTION-DIFFUSION EQUATIONS

CHACE COVINGTON, VICTORIA CHEBOTAEVA, CHUNYAN LI, ANDREI MEDVED, AND MURRAY MACNAMARA

ABSTRACT. In this report, we focus on $1D$ reaction-diffusion equations with periodic domain for several different types of reaction terms including: $-u^2$, u^2, $u(1-u)$, $u(1-u)(\alpha-u)$. We show the existence, uniqueness and instant regularization of weak solution (u, u') in space $L^2([0,T]; H^1(\mathbb{T}))$, $L^2([0,T]; H^{-1}(\mathbb{T}))$ respectively. We first construct solutions u_m of certain finite-dimensional approximations to our original problems by using the Galerkin Approximation Method. Then we do L^2 energy estimates to get uniform boundedness of u_m and u_{mt}, followed by applying Banach Alaoglu Theorem and Aubin-Lions Lemma to show a subsequence of u_m converging to a weak solution of the original problem. We also show the existence and uniqueness of mild solution in Banach space $\mathcal{E}_{\mathbb{T}}$ via Banach Fixed point Theorem for the cases $-u^2,\ u^2,\ u(1-u)$.

1. Introduction

Reaction–diffusion equations are mathematical equations which correspond to several physical phenomena. The most common is the change in space and time of the concentration of one or more chemical substances: local chemical reactions in which the substances are transformed into each other, and diffusion which causes the substances to spread out over a surface in space.

Reaction–diffusion equations are naturally applied in chemistry. However, the system can also describe dynamical processes of non-chemical nature. Examples are found in biology, geology, physics (neutron diffusion theory), and ecology. Mathematically, reaction–diffusion equations take the form of semi-linear parabolic partial differential equations. They can be represented in the general form

$$u_t - \nu u_{xx} = R(u)$$

Where $u = u(x,t)$ represents the unknown vector function, ν is a diffusion coefficient, and $R(u)$ is a smooth function $R : \mathbb{R} \to \mathbb{R}$ which accounts for all local reactions[1].

2. Background

2.1. The FKPP equation.

Investigation in this field began from the papers [2] of Fisher and Kolmogorov, Petrovsky and Piskunoff and was motivated by population dynamics issues, where authors arrived at a modified diffusion equation:

$$u_t - \nu u_{xx} = u - u^2.$$

A typical solution of this equation is a propagating front separating two non-equilibrium homogeneous states, one of which ($u = 1$) is stable and another one ($u = 0$) is unstable. The interest in physics in these type of fronts was stimulated in the early 1980's by the work of G. Dee and coworkers on the theory of dendritic solidification. Examples of such fronts can be found in various physical, chemical, and biological systems.

2.2. The Zeldovich equation.

Another important example of Reaction-Diffusion equations is the Zeldovich–Frank– Kamenetsky–Equation[3], which describes flame propagation.

$$u_t - \nu u_{xx} = u(1-u)(\alpha - u), \alpha \in (0,1).$$

This model has been suggested in 1937 for the mathematical description of combustion processes. ZFK model relates to the class of nonlinear reaction diffusion models. Actually, it was one of the first models of the class mentioned, which gave non-trivial results of great importance: in particular, this model enabled the derivation of the general analytical expression for the velocity of stationary propagating plane flame front.

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3. Reaction term $R(u) = -u^2$

In this chapter, consider the following reaction-diffusion equation:

$$\begin{cases} u_t - \nu u_{xx} = -u^2 \; in \; [0,T] \times \mathbb{T} \\ u(t=0,x) = g(x) \end{cases} \tag{1}$$

where $\mathbb{T}$ denotes periodic boundary, ν is a positive constant.

3.1. Weak Formulation and Weak Solution.

Definition 3.1. *By using test functions and divergence theorem/integration by parts, we get the weak formulation of* (1) *as follows:*
Find weak solution $u \in L^2([0,T]; H^1(\mathbb{T}))$ *such that*

$$\langle u_t, v\rangle + \nu((u,v)) = -\langle u^2, v\rangle \tag{2}$$

for all test functions $v(t,x) \in H^1(\mathbb{T})$ *where* $\langle\cdot,\cdot\rangle$ *denotes* L^2 *inner product, i.e.* $\langle u_t, v\rangle := \int_{\mathbb{T}} u_t v dx$, $((\cdot,\cdot))$ *denotes* H^1 *inner product, i.e.* $((u,v)) := \int_{\mathbb{T}} u_x v_x dx$ *and* $\langle u^2, v\rangle := \int_{\mathbb{T}} u^2 v dx$

Note: u_t denotes the weak time derivative of u with respect to the initial condition with u_t satisfying

$$\int_{\mathbb{T}}\int_0^T u_t v dt dx = -v(t=0,x)u(t=0,x) - \int_{\mathbb{T}} u v_t dt dx$$

3.2. Galerkin Approximation Method.

We first use Galerkin approximation method to get an approximated solution of (1) in finite dimensional space. Choose finite dimensional subspace $V_m = span\{v_1, v_2, ..., v_m\} \subseteq H^1(\mathbb{T})$ to find the approximation solution $u_m(t,\cdot) \in V_m$ such that

$$(u_{mt}, v) + \nu((u_m, v)) = -\mathbb{P}[(u_m^2, v)] \tag{3}$$

for all test functions $v \in V_m$. Where $\mathbb{P}$ is the projection mapping such that $\mathbb{P} : \mathbb{P}[\sum_{i=1}^{\infty} a_i v_i] = \sum_{i=1}^{m} a_i v_i$.

Let's assume $u_m = \sum_{j=0}^{m} c_j(t) v_j(x)$. Then, one can get

$$\sum_{j=1}^{m} c_j'(t)(v_j, v_i) + \nu \sum_{j=1}^{m} c_j(t)((v_j, v_i)) = -\mathbb{P}[((\sum_{j=1}^{m} c_t v_j)^2, v_i)] \tag{4}$$

for all $i = 1, 2, ..., m$.
Let's assume that basis functions v_i have L^2-norm and H^1-norm orthogonality. We can orthonormalize basis in L^2-norm and simplify (4), so that an ODE system is obtained:

$$\begin{cases} I \cdot C'(t) + \nu B \cdot C(t) + \mathbb{P}[F(t, C(t))] = 0 \\ C(0) = \mathbf{g} \end{cases} \tag{5}$$

Where $C(t) = (c_1(t), ..., c_m(t))'$, matrix $B = (b_{ij})_{m\times m}$ with $b_{ij} = ((v_j, v_i))$, If $\mathbb{P}[F_C(t,C)]$ is bounded, one can get existence and uniqueness of solution $u_m \in V_m$ from the theory of ODE.
We propose now to send m to infinity and show a subsequence of our solutions u_m which converges to a weak solution of (1) by doing some uniform estimate. In order to simplify the notation,we denote u_m by u in the following sections.

3.3. Existence.

Theorem 3.2. ***Weak compactness-special case of Banach-Alaoglu Theorem***
Let X *be a reflexive Banach space and suppose the sequence* $\{u_k\}_{k=1}^{\infty} \subset X$ *is bounded. Then there exists a subsequence* $\{u_{k_j}\}_{j=1}^{\infty} \subset \{u_k\}_{k=1}^{\infty}$ *and* $u \in X$ *such that*

$$u_{k_j} \rightharpoonup u$$

One can use Banach-Alaoglu Theorem to get a candidate solution of (1).

3.3.1. *Uniform boundedness of* $\max_{0\le t\le T}||u||^2_{L^2}$.

Choose $v = u_m \in V_m$ as test function in (3),

$$\int uu_t dx + \nu \int u_x^2 dx = -\int u^3 dx \le 0$$
$$\Rightarrow \frac{d}{dt}\int \frac{1}{2}u^2 dx + \nu \int u_x^2 dx \le 0$$

Define $E(t) = \int u^2 dx$

$$E'(t) + 2\nu \int u_x^2 dx \le 0$$
$$\Rightarrow E'(t) \le 0$$
$$\Rightarrow E(t) \le E(0) = ||g||^2_{L^2} \quad \forall t \in [0, T]$$

Hence,

$$\int u^2 dx \le ||g||^2_{L^2}$$
$$\Rightarrow \max_{0\le t\le T} ||u||^2_{L^2} \le ||g||^2_{L^2}.$$

3.3.2. *Uniform boundedness of* u_m *in* L^2H^1 *norm.*

$$E'(t) + 2\nu \int u_x^2 dx + 2\nu \int u^2 dx \le 0 + 2\nu \int u^2 dx$$
$$\Rightarrow E'(t) + 2\nu||u^2||_{H^1} \le 2\nu E(t)$$
$$E'(t) \le -2\nu||u||^2_{H^1} + 2\nu E(t)$$

By Grönwall inequality:

$$E(t) \le e^{\int_0^t 2\nu ds}\left(||g||^2_{L^2} + \int_0^t -2\nu||u(s,\cdot)||^2_{H^1} ds\right)$$
$$\Rightarrow e^{2\nu t} 2\nu \int_0^t ||u(s,\cdot)||^2_{H^1} ds \le e^{\int_0^t 2\nu ds}||g||^2_{L^2} - E(t) \le e^{2\nu t}||g||^2_{L^2}$$
$$||u(t,x)||_{L^2H^1} \le \frac{||g||^2_{L^2}}{2\nu}$$

Since u_m is uniformly bounded in L^2H^1, by Banach-Alaoglu theorem, u_m converges weakly to a limit point u in L^2H^1 up to a subsequence.

3.3.3. *Uniform boundedness of* u_{m_t} *in* L^1H^{-1}.

By the decomposition of Hilbert space $H^1(\mathbb{T})$ which is $H^1(\mathbb{T}) = V_m + V_m^\perp$, one has $\forall v \in H^1(\mathbb{T})$, $v = v_1 + v_2$ where $v_1 \in V_m, v_2 \in V_m^\perp$.
By the orthogonality, we have

$$\langle u_t, v\rangle = \langle u_t, v_1 + v_2\rangle = \langle u_t, v_1\rangle + \langle u_t, v_2\rangle = \langle u_t, v_1\rangle$$

From the weak formulation (3) we get:

$$\langle u_t, v_1\rangle = -\nu((u, v_1)) - (u^2, v_1)$$

Hence,

$$\langle u_t, v\rangle = -\nu((u, v_1)) - (u^2, v_1)\ \forall v \in H^1(\mathbb{T}).$$

Then,

$$\begin{aligned}
|\langle u_t, v\rangle| &\le \nu|((u, v_1))| + |(u^2, v_1)| \\
&\le \nu||u||_{H^1}||v_1||_{H^1} + ||u^2||_{L^2}||v_1||_{L^2} \\
&\le \nu||u||_{H^1}||v||_{H^1} + ||u^2||_{L^2}||v||_{H^1} \\
\Rightarrow \frac{|\langle u_t, v\rangle|}{||v||_{H^1}} &\le \nu||u||_{H^1} + ||u^2||_{L^2}
\end{aligned}$$

By the definition $||u_t||_{H^{-1}} = \sup_{v\in H^1(\mathbb{T})} \dfrac{|\langle u_t, v\rangle|}{||v||_{H^1}}$, we have

$$||u_t(t,\cdot)||_{H^{-1}} \le \nu||u(t,\cdot)||_{H^1} + ||u(t,\cdot)||^2_{L^4}$$

Take L^1 integral over $[0,T]$

$$\int_0^T ||u_t(s,\cdot)||_{H^{-1}}ds \leq \nu\int_0^T ||u(s,\cdot)||_{H^1}ds + \int_0^T ||u(s,\cdot)||_{L^4}^2 ds$$

$$\Rightarrow ||u_t||_{L^1H^{-1}} \leq \nu||u||_{L^1H^1} + ||u||_{L^2L^4}$$

By Sobolev embedding $H^1 \hookrightarrow L^4$, we get

$$||u||_{L^4} \leq C||u||_{H^1}$$

$$\Rightarrow ||u||_{L^2L^4} \leq C||u||_{L^2H^1}$$

By Hölder inequality, we get

$$||u||_{L^1H^1} \leq \sqrt{T}||u||_{L^2H^1}$$

Hence, we get

$$||u_t||_{L^1H^{-1}} \leq \nu\sqrt{T}||u||_{L^2H^1} + C||u||_{L^2H^1}$$

$$\leq (\nu\sqrt{T}+C)||u||_{L^2H^1}$$

$$\leq \widetilde{C}||g||_{L^2}^2$$

Since u_{mt} is uniformly bounded in L^1H^{-1}, by Banach-Alaoglu theorem, u_{mt} converges weakly to a limit point u' in $\mathcal{M}H^{-1}$ up to a subsequence. One can extend this result to L^2H^{-1}.

3.3.4. *Extended Uniform boundedness u_{mt} in $L^2([0,T];H^{-1}(\mathbb{T}))$.*

Let's focus on the estimate of $||u_t(t,\cdot)||_{H^{-1}}$ that we get in previous subsection.

$$||u_t(t,\cdot)||_{H^{-1}} \leq \nu||u(t,\cdot)||_{H^1} + ||u(t,\cdot)||_{L^4}^2$$

Take L^2 integral over $[0,T]$

$$\int_0^T ||u_t(s,\cdot)||_{H^{-1}}^2 ds \leq \nu^2\int_0^T ||u(s,\cdot)||_{H^1}^2 ds + \int_0^T ||u(s,\cdot)||_{L^4}^4 ds + \int_0^T 2\nu||u||_{H^1}||u||_{L^4}^2 ds$$

$$\Rightarrow ||u_t||_{L^2H^{-1}} \leq \left(\nu^2 + 2M\nu\right)||u||_{L^2H^1} + M^2||u||_{L^2L^2}$$

Where the last inequality is valid due to $||u||_{L^4}^4 = \int u^4dx \leq M^2\int u^2dx = M^2||u||_{L^2}^2$ and $||u||_{L^4}^2 = \left(\int u^4dx\right)^{1/2} \leq M\left(\int u^2dx\right)^{1/2} = M||u||_{L^2} \leq M||u||_{H^1}$ by $0 \leq u \leq M$ which will be proved in the later section.Hence, $||u_{mt}||_{L^2H^{-1}}$ is uniformly bounded by the uniform boundedness of $||u||_{L^2H^1}$ and $\max_{0\leq t\leq T}||u||_{L^2}^2$.

3.3.5. *Passing the limit to the non-linear term.*

Theorem 3.3. ***Aubin-Lions Lemma*** *Let X_0, X, X_1 be three Banach spaces with $X_0 \subseteq X \subseteq X_1$. Suppose that X_0 is compactly embedded in X and that X is continuously embedded in X_1. For $1 \leq p,q \leq \infty$, let*

$$W = \{u \in L^p([0,T];X_0) | u' \in L^q([0,T];X_1)\}$$

- *If $p < \infty$, then the embedding of W into $L^p([0,T];X)$ is compact.*
- *If $p = \infty$ and $q > 1$, then the embedding of W into $C([0,T];X)$ is compact.*

Because $H^1 \hookrightarrow L^2 \hookrightarrow H^{-1}$ with $H^1 \hookrightarrow L^2$ being a compact embedding and $L^2 \hookrightarrow H^{-1}$ being a continuous embedding, Aubin-Lions Lemma tells us that $u_{m_j} \to u$ in $L^2([0,T],L^2(\mathbb{T}))$. Then we have $\lim_{j\to\infty}\int u_{m_j}^2 = \int u^2dx$. Then $\lim_{j\to\infty}\int_{\mathbb{T}} u_{m_j}^2 vdx = \int_{\mathbb{T}} u^2vdx$ for any function $v \in V_m$.

3.3.6. *Conclusion.* Summarize the results above, we have

$$\begin{aligned} \left(u_{m_j t}, v\right) &\to (u_t, v) \\ \left(\left(u_{m_j}, v\right)\right) &\to ((u, v)) \\ \left(u_{m_j}^2, v\right) &\to \left(u^2, v\right) \end{aligned}$$

for $v \in V_m$.
By property of Hilbert space, we know that $V_m \to H^1(\mathbb{T})$ as $m \to \infty$.
Thus,

$$\begin{aligned} \lim_{j\to\infty} \left(\left(u_{m_j t}, v\right) + \nu\left(\left(u_{m_j}, v\right)\right)\right) &= \lim_{j\to\infty} \left(-u_{m_j}^2, v\right) \;\; for\ v \in V_m \\ \langle u_t, v\rangle + \nu((u, v)) &= -\left\langle u^2, v\right\rangle \;\; for\ v \in H^1 \end{aligned}$$

One can easily check that u', the limit point of $u_{m_{jt}}$, satisfies the initial condition. Thus, u is a weak solution in $L^2([0,T]; H^1(\mathbb{T}))$. Up to now we get the existence of a weak solution, and in the following sections we will focus on the uniqueness of our weak solution.

3.4. **Uniqueness.**

Let $u_1, u_2 \in L^2([0,T]; H^1(\mathbb{R}))$ be two weak solutions of

$$\begin{cases} u_t - \nu u_{xx} = -u^2 \\ u(t=0,x) = g(x) \end{cases} \tag{6}$$

Then $w = u_1 - u_2$ satisfies

$$\begin{cases} w_t - \nu w_{xx} = u_2^2 - u_1^2 \\ w(t=0,x) = 0 \end{cases} \tag{7}$$

Note that $u_2^2 - u_1^2 = -w(u_1 + u_2)$, allowing (7) to written as

$$\begin{cases} w_t - \nu w_{xx} = -w\left(u_1 + u_2\right) \\ w(t=0,x) = 0 \end{cases} \tag{8}$$

The weak formulation associated with (8) is

$$(w_t, v) + \nu\left((w, v)\right) = \left(-w\left(u_1 + u_2\right), v\right) \tag{9}$$

for $v \in V_m$ and where $(f,g) = \int_{\mathbb{T}} fg dx$ and $((f,g)) = \int_{\mathbb{T}} f_x g_x dx$. Now choose $v = w$. Assume that $u_1, u_2 \geq 0$. Then

$$\begin{aligned} (w_t, w) + \nu\left((w, w)\right) &= \left(-w\left(u_1 + u_2\right), w\right) \\ \int_{\mathbb{T}} w w_t dx + \nu \int_{\mathbb{T}} w_x^2 dx &= -\int_{\mathbb{T}} w^2\left(u_1 + u_2\right) dx \\ \frac{d}{dt}\int_{\mathbb{T}} \frac{1}{2} w^2 dx + \nu \int_{\mathbb{T}} w_x^2 dx &= -\int_{\mathbb{T}} w^2\left(u_1 + u_2\right) dx \\ &\leq 0 \end{aligned}$$

Note that $\int_{\mathbb{T}} w_x^2 dx + \int_{\mathbb{T}} w^2 dx = ||w||_{H^1}^2$. Let

$$E(t) := \int_{\mathbb{T}} \frac{1}{2} w^2(t, \cdot) dx \tag{10}$$

be the total energy at time t. Then

$$\begin{aligned} E'(t) + \nu \int_{\mathbb{T}} w_x^2 dx &\leq 0 \\ E'(t) + \nu \int_{\mathbb{T}} w_x^2 dx + \nu \int_{\mathbb{T}} w^2 dx &\leq \nu \int_{\mathbb{T}} w^2 dx \\ E'(t) + \nu ||w||_{H^1}^2 &\leq 2\nu E(t) \end{aligned}$$

Applying Grönwall's Inequality yields

$$E(t) \leq e^{\int_0^t 2\nu ds}\left(E(0) - \int_0^t \nu ||w(s,\cdot)||_{H^1}^2 ds\right) \tag{11}$$

Note that $\int_0^t \nu ||w^2(s,\cdot)||_{H^1}^2 ds \geq 0$ and $E(0) = 0$, so

$$E(t) \leq 0 \tag{12}$$

Note that for $E(t) = \int_{\mathbb{T}} w^2(t,x)dx \geq 0$, $E(0) = 0$. Thus, $w \equiv 0$ *a.e.*, i.e. $u_{m1} = u_{m2}$ a.e for all m. Then assume u is bounded, which will be verified in the last section. By Dominated convergence theorem, we have

$$\lim_{m\to\infty} \int\int (u_{m1} - u_{m2})vdxdt = \int\int (u_1 - u_2)vdxdt = 0$$

for all v, hence, $u_1 = u_2$ a.e. and uniqueness follows.

3.5. Mild Solution.

Definition 3.4. ***Mild Solution***
For the reaction-diffusion equation

$$\begin{cases} u_t - \nu u_{xx} = -u^2 \text{ in } [0,T] \times \mathbb{T} \\ u(t=0,x) = g(x) \end{cases}$$

the solution satisfying:

$$u(t,x) = S(t)g + \int_0^t [-S(t-s)u^2(s)]ds \tag{13}$$

is called the mild solution. $S(t)$ is a semi-group defined as follows:

$$S(t,\cdot)g(\cdot) = \Phi(t,\cdot) * g(\cdot) = \int_{\mathbb{R}} \frac{1}{\sqrt{4t\nu\pi}} e^{\frac{-|x-y|^2}{4t\nu}} g(y)dy$$

with $\Phi(t,x) = \frac{1}{\sqrt{4t\nu\pi}} e^{\frac{-|x|^2}{4t\nu}}$ being called the heat kernel for the whole domain. In order to keep consistency with periodic domain, one call also write the periodic heat kernel $\Phi_{\mathbb{T}}(t,x) = \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{4t\pi\nu}} e^{-\frac{(y-k)^2}{4t\nu}}$, then, the simi-group is defined as follows: $S(t,\cdot)g(\cdot) = \int_{\mathbb{T}} \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{4t\pi\nu}} e^{-\frac{(y-k)^2}{4t\nu}} g(x-y)dy$.

Note: This kind of mild solution is a solution depending on itself, and as such must be treated iteratively when refering to a numerical approach.

Theorem 3.5. ***Banach Fixed Point Theorem*** *Let X be a Banach space and $T : X \to X$ a linear operator. Suppose $\exists\ q \in [0,1)$ such that $\forall x, y \in X$, $||Tx - Ty||_X \leq q||x - y||_X$, then T has a unique fixed point with $Tx = x$.*

3.5.1. *Application of Banach Fixed Point Theorem.* In order to apply this theorem, we define the following mapping:

$$\begin{aligned} T &: X \to X \\ T(u) &= S(t)g(x) - B[u,u] \end{aligned} \tag{14}$$

where $B[u,v] = \int_0^t S(t-s)\left(u(s,y)v(s,y)\right)ds$, and X being our solution space which we are free to pick. We will justify that T maps X into itself later.

3.5.2. *Picking the Banach space X.* Choose $X = \mathcal{E}_{\mathbb{T}} = \{u \in L^\infty([0,T]; L^p(\mathbb{T})) : ||u||_{\mathcal{E}_{\mathbb{T}}} \leq \delta\}$ with $||u||_{\mathcal{E}_{\mathbb{T}}} := ||u||_{L^\infty L^p}$, and δ a small positive constant.

3.5.3. *$T(u) \in \mathcal{E}_{\mathbb{T}}$.* Let us first make the assumption

$$||B[u,v]||_{\mathcal{E}_{\mathbb{T}}} \leq c||u||_{\mathcal{E}_{\mathbb{T}}}||v||_{\mathcal{E}_{\mathbb{T}}}$$

for some constant c.
From this we have

$$\begin{aligned} ||T(u)||_{\mathcal{E}_{\mathbb{T}}} &\leq ||S(t)g(x)||_{\mathcal{E}_{\mathbb{T}}} + ||B[u,u]||_{\mathcal{E}_{\mathbb{T}}} \\ &\leq ||S(t)g(x)||_{\mathcal{E}_{\mathbb{T}}} + c||u||^2_{\mathcal{E}_{\mathbb{T}}} \end{aligned}$$

By the choice of $\mathcal{E}_{\mathbb{T}}$ we then have

$$\begin{aligned} ||T(u)||_{\mathcal{E}_{\mathbb{T}}} &\leq ||S(t)g(x)||_{\mathcal{E}_{\mathbb{T}}} + c\delta^2 \\ &\leq ||g(x)||_{L^p} + c\delta^2 \end{aligned}$$

where the last inequality follows from the Young's inequality for convolution and $||\Phi(t,\cdot)||_{L^1} = 1\ \forall t$.
If we require $||g(x)||_{L^p} \leq \frac{\delta}{2}$ and $c\delta^2 \leq \frac{\delta}{2}$, we then have $T(u) : \mathcal{E}_{\mathbb{T}} \to \mathcal{E}_{\mathbb{T}}$ with $\delta < \frac{1}{2c}$.

3.5.4. *Checking T is a Contraction Mapping.* In this part, we show that T is a contraction mapping, i.e. proving that T satisfies

$$||T(u) - T(v)||_{\mathcal{E}_{\mathbb{T}}} \leq q||u - v||_{\mathcal{E}_{\mathbb{T}}}$$

where $q \in [0, 1)$. From (14) we have

$$\begin{aligned} ||T(u) - T(v)||_{\mathcal{E}_{\mathbb{T}}} &= ||B[u, u] - B[v, v]||_{\mathcal{E}_{\mathbb{T}}} \\ &= ||B[u, u - v] + B[v, u - v]||_{\mathcal{E}_{\mathbb{T}}} \end{aligned}$$

From our previous assumption we then have

$$\begin{aligned} ||T(u) - T(v)||_{\mathcal{E}_{\mathbb{T}}} &\leq c(||u||_{\mathcal{E}_{\mathbb{T}}}||u - v||_{\mathcal{E}_{\mathbb{T}}} + ||v||_{\mathcal{E}_{\mathbb{T}}}||u - v||_{\mathcal{E}_{\mathbb{T}}}) \\ &\leq c||u - v||_{\mathcal{E}_{\mathbb{T}}}(||u||_{\mathcal{E}_{\mathbb{T}}} + ||v||_{\mathcal{E}_{\mathbb{T}}}) \end{aligned}$$

With the choice of the space $\mathcal{E}_{\mathbb{T}}$ we then have

$$||T(u) - T(v)||_{\mathcal{E}_{\mathbb{T}}} \leq 2\delta c(||u - v||_{\mathcal{E}_{\mathbb{T}}})$$

Choose $\delta < \frac{1}{2c}$ such that

$$||T(u) - T(v)||_{\mathcal{E}_{\mathbb{T}}} < ||u - v||_{\mathcal{E}_{\mathbb{T}}}$$

So with sufficiently small initial data, i.e. $||g(x)||_{L^p} \leq \frac{\delta}{2} < \frac{1}{4c}$ combined with the assumption of $||B[u, v]||_{\mathcal{E}_{\mathbb{T}}}$, then $\exists!$ $u(x, t) \in \mathcal{E}_{\mathbb{T}}$ that satisfies (14) by the Banach Fixed Point Theorem.

3.5.5. *Verify the assumption of $B[u, v]$.* We have

$$\begin{aligned} B[u, v] &= \int_0^t \Phi(t - s, \cdot) * (u(s, \cdot)v(s, \cdot))ds \\ &= \int_0^t \int_{\mathbb{T}} \Phi(t - s, x - y)u(s, y)v(s, y)dyds \\ &= \int_0^t \frac{1}{\sqrt{4(t - s)\nu\pi}} \int_{\mathbb{T}} e^{\frac{-|x-y|^2}{4(t-s)\nu}} u(s, y)v(s, y)dyds \end{aligned} \tag{15}$$

By Minkowski's inequality for integrals,

$$||B[u, v]||_{L^p} \leq \int_0^t ||\Phi(t - s, \cdot) * (u(s, \cdot)v(s, \cdot))||_{L^p} ds$$

By Young's Convolution Inequality, Hölder's inequality, and fixing $t \in [0, T]$ we then have

$$||B[u, v]||_{L^p} \leq \int_0^t \frac{1}{\sqrt{4(t - s)\nu\pi}} ||e^{\frac{-x^2}{4(t-s)\nu}}||_{L^r} ||u(s, \cdot)||_{L^p} ||v(s, \cdot)||_{L^p} ds$$

where $1 + \dfrac{1}{p} = \dfrac{1}{r} + \dfrac{1}{p} + \dfrac{1}{p}$, namely, $1 = \dfrac{1}{r} + \dfrac{1}{p}$
By taking the essential sup in time we get

$$||B[u, v]||_{\mathcal{E}_{\mathbb{T}}} \leq \int_0^t \frac{1}{\sqrt{4(t - s)\nu\pi}} ||e^{\frac{-x^2}{4(t-s)\nu}}||_{L^r} ds ||u||_{\mathcal{E}_{\mathbb{T}}} ||v||_{\mathcal{E}_{\mathbb{T}}}$$

In order to show that

$$||B[u, v]||_{\mathcal{E}_{\mathbb{T}}} \leq c||u||_{\mathcal{E}_{\mathbb{T}}} ||v||_{\mathcal{E}_{\mathbb{T}}}$$

we need now only prove that $\int_0^t ||\Phi(t - s, \cdot)||_{L^r} ds$ is finite for any time $t \in [0, T]$. Note

$$\begin{aligned} \left(\int_{\mathbb{T}} \left(e^{\frac{-x^2}{4\nu(t-s)}} \right)^r dx \right)^{\frac{1}{r}} &= \left(\int_{\mathbb{T}} e^{\frac{-rx^2}{4\nu(t-s)}} dx \right)^{\frac{1}{r}} \\ &= \left(\frac{4\pi\nu(t - s)}{r} \right)^{\frac{1}{2r}} \end{aligned}$$

Absorbing all constants into α we now have

$$\begin{aligned} \frac{||B[u, u]||_{\mathcal{E}_{\mathbb{T}}}}{||u||^2_{\mathcal{E}_{\mathbb{T}}}} &\leq \alpha \int_0^t (t - s)^{\frac{-1}{2}} (t - s)^{\frac{1}{2r}} ds \\ &\leq \alpha \int_0^t (t - s)^{\frac{1}{2}(\frac{1}{r} - 1)} ds \\ &\leq \beta (t - s)^{\frac{1}{2r} + \frac{1}{2}} |_0^t \end{aligned}$$

Where β has absorbed the integration constant. For this value to be finite for any time t, we need $\frac{1}{2r} + \frac{1}{2} \geq 0$, thus $r = 1 - \frac{1}{p} \geq 0$ satisfies.

3.5.6. *Conclusion.* To summarize, we have proven that we can use the Banach Fixed Point Theorem combined with the assumption

$$||B[u,v]||_{\mathcal{E}_\mathbb{T}} \le c||u||_{\mathcal{E}_\mathbb{T}}||v||_{\mathcal{E}_\mathbb{T}}$$

to find a unique mild solution u(t,x) for the reaction diffusion equation with the reaction term R(u)=-u^2. As this solution depends on itself we are then able to use this iterative process

$$\begin{cases} u_{k+1} = Tu_k \\ u_0 = g(x) \end{cases}$$

for our final solution.
Due to the method of this procedure this result can be extended to the reaction term R(u)=u^2 as well.

4. **Reaction term** $R(u) = u(1-u)$

4.1. **Existence.** The weak formulation is

$$\langle u_t, v\rangle + \nu((u,v)) = \langle u(1-u), v\rangle \;\; \forall v \in H^1(\mathbb{T}) \tag{16}$$

4.1.1. *Uniform boundedness of* $\max_{0\le t\le T} ||u||^2_{L^2}$.

Choose $v = u_m \in V$ as test function in (4), assume $u \ge 0$ by its physical meaning,

$$\int u_t u dx + \nu \int u_x^2 dx = \int u^2(1-u)dx$$

$$\frac{d}{dt}\int \frac{1}{2}u^2 dx + \nu\int u_x^2 dx = \int u^2 - u^3 dx \le \int u^2 dx$$

Define $E(t) = \int u^2 dx$

$$\frac{d}{dt}\int u^2 dx \le 2\int u^2 dx$$

$$\Rightarrow E'(t) \le 2E(t)$$

By Grönwall inequality,

$$\Rightarrow E(t) \le e^{2t}E(0) = e^{2t}||g||^2_{L^2} \;\; \forall t \in [0,T]$$

Hence,

$$\int u^2 dx \le e^{2T}||g||^2_{L^2}$$

$$\Rightarrow \max_{0\le t\le T} ||u||^2_{L^2} \le e^{2T}||g||^2_{L^2}.$$

4.1.2. *Uniform boundedness of* u_m *in* $L^2([0,T]; H^1(\mathbb{T}))$.

Let test function $v = u_m \in V_m$

$$\int u_t u dx + \nu \int u_x^2 dx = \int u^2(1-u)dx$$

$$\frac{d}{dt}\int \frac{1}{2}u^2 dx + \nu\int u_x^2 dx = \int u^2 - u^3 dx \le \int u^2 dx$$

Define $E(t) = \int u^2 dx$

$$\frac{d}{dt}\int u^2 dx + 2\nu\int u_x^2 dx + 2\nu\int u^2 dx \le 2\int u^2 dx + 2\nu\int u^2 dx$$

$$\Rightarrow E'(t) + 2\nu||u||^2_{H^1} \le (2+2\nu)E(t)$$

By Grönwall inequality,

$$E(t) \le e^{\int_0^t 2+2\nu ds}\left(E(0) + \int_0^t -2\nu||u(s,\cdot)||^2_{H^1} ds\right)$$

Solve for $||u||_{L^2H^1}$ to get

$$e^{(2+2\nu)t}2\nu||u||_{L^2H^1} \le e^{(2+2\nu)t}E(0) - E(t)$$

$$\Rightarrow ||u||_{L^2H^1} \leq \frac{E(0)}{2\nu} = \frac{||g||_{L^2}^2}{2\nu}$$

4.1.3. *Uniform boundedness* u_{m_t} *in* $L^1([0,T];H^{-1}(\mathbb{T}))$.

Let $v_1 \in V_m$, by the same decomposition of $H^1(\mathbb{T})$ in previous section, we get

$$\langle u_t, v\rangle = -\nu((u,v_1)) + (u,v_1) - (u^2,v_1) \ \forall v \in H^1(\mathbb{T}).$$

Then,

$$\begin{aligned} |\langle u_t, v\rangle| &\leq \nu|((u,v_1))| + |(u,v_1)| + |(u^2,v_1)| \\ &\leq \nu||u||_{H^1}||v_1||_{H^1} + ||u||_{L^2}||v_1||_{L^2} + ||u^2||_{L^2}||v_1||_{L^2} \\ &\leq \nu||u||_{H^1}||v||_{H^1} + ||u||_{H^1}||v||_{H^1} + ||u^2||_{L^2}||v||_{H^1} \\ \Rightarrow \frac{|\langle u_t, v\rangle|}{||v||_{H^1}} &\leq (\nu+1)||u||_{H^1} + ||u^2||_{L^2} \end{aligned}$$

By the definition $||u_t||_{H^{-1}} = \sup_{v\in H^1(\mathbb{T})} \dfrac{|\langle u_t, v\rangle|}{||v||_{H^1}}$, we have

$$||u_t(t,\cdot)||_{H^{-1}} \leq (\nu+1)||u(t,\cdot)||_{H^1} + ||u(t,\cdot)||_{L^4}^2 \tag{17}$$

Take L^1 integral over $[0,T]$

$$\int_0^T ||u_t(s,\cdot)||_{H^{-1}} ds \leq (\nu+1)\int_0^T ||u(s,\cdot)||_{H^1} ds + \int_0^T ||u(s,\cdot)||_{L^4}^2 ds$$

$$\Rightarrow ||u_t||_{L^1H^{-1}} \leq (\nu+1)||u||_{L^1H^1} + ||u||_{L^2L^4}$$

By Sobolev embedding $H^1 \hookrightarrow L^4$, we get

$$||u||_{L^4} \leq C||u||_{H^1}$$
$$\Rightarrow ||u||_{L^2L^4} \leq C||u||_{L^2H^1}$$

By Hölder inequality, we get

$$||u||_{L^1H^1} \leq \sqrt{T}||u||_{L^2H^1}$$

Hence, we get

$$\begin{aligned} ||u_t||_{L^1H^{-1}} &\leq (\nu+1)\sqrt{T}||u||_{L^2H^1} + C||u||_{L^2H^1} \\ &\leq ((\nu+1)\sqrt{T} + C)||u||_{L^2H^1} \\ &\leq \widetilde{C}||g||_{L^2}^2 \end{aligned}$$

Since u_{mt} is uniformly bounded in L^1H^{-1}, by Banach-Alaoglu theorem, u_{mt} converges weakly to a limit point u' in $\mathcal{M}H^{-1}$ up to a subsequence.

4.1.4. *Extended Uniform boundedness* u_{mt} *in* $L^2([0,T];H^{-1}(\mathbb{T}))$.

First, consider the following ode problem:

$$\begin{cases} u_t = u(1-u) \ in \ [0,T]\times\mathbb{T} \\ u(t=0,x) = b \ where \ 0 \leq b \leq 1 \end{cases} \tag{18}$$

One can easily check that $u = 0, 1$ are two steady state solutions of this ODE. And $u = 1$ is the only stable solution. Then by the ODE theory, we know that if initial data b lies in $[0,1]$, our solution u will also lie in $[0,1]$. The diffusion term doesn't destroy this structure, hence, in this section, we assume the solution of the corresponding reaction diffusion equation lies in $[0,1]$ with initial data provided in $[0,1]$.
Now, Let's focus on (17)

$$||u_t(t,\cdot)||_{H^{-1}} \leq (\nu+1)||u(t,\cdot)||_{H^1} + ||u(t,\cdot)||_{L^4}^2$$

Take L^2 integral over $[0,T]$

$$\int_0^T ||u_t(s,\cdot)||_{H^{-1}}^2 ds \leq (\nu+1)^2\int_0^T ||u(s,\cdot)||_{H^1}^2 ds + \int_0^T ||u(s,\cdot)||_{L^4}^4 ds + \int_0^T 2(\nu+1)||u||_{H^1}||u||_{L^4}^2 ds$$

$$\Rightarrow ||u_t||_{L^2H^{-1}} \leq \left((\nu+1)^2 + 2(\nu+1)\right)||u||_{L^2H^1} + ||u||_{L^2L^2}$$

Where the last inequality is valid due to $||u||_{L^4}^4 = \int u^4 dx \leq \int u^2 dx = ||u||_{L^2}^2$ and $||u||_{L^4}^2 = \left(\int u^4 dx\right)^{1/2} \leq \left(\int u^2 dx\right)^{1/2} = ||u||_{L^2} \leq ||u||_{H^1}$ by $0 \leq u \leq 1$.Hence, $||u_{mt}||_{L^2H^{-1}}$ is uniformly bounded by the uniform boundedness of $||u||_{L^2H^1}$ and $\max_{0\leq t\leq T} ||u||_{L^2}^2$.

4.2. **Passing the limit to the non-linear term.**

$$(u(1-u), v) = (u, v) - (u^2, v)$$

By the uniform boundedness of $\max_{0\le t\le T} ||u||_{L^2}^2$ and Banach-Alaoglu Theorem, we know that

$$(u_m, v) \to (u, v)$$

As the same procedure in 3.3.5 by Aubin-Lions Lemma, we know that

$$\left(u_m^2, v\right) \to \left(u^2, v\right)$$

Thus, combined with the results in previous section 4.1, we get

$$\begin{aligned}
\lim_{j\to\infty}\left(\left(u_{m_j t}, v\right) + \nu\left(\left(u_{m_j}, v\right)\right)\right) &= \lim_{j\to\infty}\left(u_{m_j}\left(1-u_{m_j}\right), v\right) \\
&= \lim_{j\to\infty}\left(\left(u_{m_j}, v\right) - \left(u_{m_j}^2, v\right)\right) \\
\langle u_t, v\rangle + \nu((u, v)) &= (u, v) - \left(u^2, v\right) \\
&= \langle (u(1-u), v\rangle
\end{aligned}$$

for $v \in H_0^1$. One can easily check that u', the limit point of $u_{m_j t}$, satisfies the initial condition. Thus, by the same procedure in 3.3.6, u is a weak solution in $L^2([0,T]; H^1(\mathbb{T}))$.

4.3. **Uniqueness.**

Let $u_1, u_2 \in L^2([0,T]; H^1(\mathbb{T}))$ be two weak solutions of

$$\begin{cases} u_t - \nu u_{xx} = u(1-u) \\ u(t=0, x) = g(x) \end{cases} \tag{19}$$

Then $w = u_1 - u_2$ satisfies

$$\begin{cases} w_t - \nu w_{xx} = u_1\left(1-u_1\right) - u_2\left(1-u_2\right) \\ w(t=0, x) = 0 \end{cases} \tag{20}$$

Note that

$$\begin{aligned}
u_1\left(1-u_1\right) - u_2\left(1-u_2\right) &= u_1 - u_2 + u_2^2 - u_1^2 \\
&= w - w\left(u_1 + u_2\right)
\end{aligned}$$

Then (20) can be written as

$$\begin{cases} w_t - \nu w_{xx} = w - w\left(u_1 + u_2\right) \\ w(t=0, x) = 0 \end{cases} \tag{21}$$

The weak formulation associated with (21) is

$$\left(w_t, v\right) + \nu\left((w, v)\right) = \left(w - w\left(u_1 + u_2\right), v\right) \tag{22}$$

for $v \in V_m$ and where $(f, g) = \int_{\mathbb{T}} fg dx$ and $((f, g)) = \int_{\mathbb{T}} f_x g_x dx$. Now choose $v = w$. Assume that $u_1, u_2 \ge 0$. Then

$$\begin{aligned}
\left(w_t, w\right) + \nu\left((w, w)\right) &= \left(w - w\left(u_1 + u_2\right), w\right) \\
\int_{\mathbb{T}} w w_t dx + \nu \int_{\mathbb{T}} w_x^2 dx &= \int_{\mathbb{T}} \left(w^2 - w^2\left(u_1 + u_2\right)\right) dx \\
\frac{d}{dt}\int_{\mathbb{T}} \frac{1}{2} w^2 dx + \nu \int_{\mathbb{T}} w_x^2 dx &= \int_{\mathbb{T}} \left(w^2 - w^2\left(u_1 + u_2\right)\right) dx \\
&\le \int_{\mathbb{T}} w^2 dx
\end{aligned}$$

Note that $\int_{\mathbb{T}} w_x^2 dx + \int_{\mathbb{T}} w^2 dx = ||w||_{H^1}^2$. Let

$$E(t) := \int_{\mathbb{T}} \frac{1}{2} w^2(t, \cdot) dx \tag{23}$$

be the total energy at time t. Then

$$\begin{aligned}
E'(t) + \nu \int_{\mathbb{T}} w_x^2 dx &\le \int_{\mathbb{T}} w^2 dx \\
E'(t) + \nu \int_{\mathbb{T}} w_x^2 dx + \nu \int_{\mathbb{T}} w^2 dx &\le \int_{\mathbb{T}} w^2 dx + \nu \int_{\mathbb{T}} w^2 dx \\
E'(t) + \nu ||w||_{H^1}^2 &\le (2\nu + 2) E(t)
\end{aligned}$$

Applying Grönwall's Inequality yields

$$E(t) \le e^{\int_0^t (2\nu+2)ds}\left(E(0) - \int_0^t \nu ||w(s,\cdot)||^2_{H^1} ds\right) \tag{24}$$

Note that $\int_0^t \nu ||w^2(s,\cdot)||^2_{H^1} ds \ge 0$ and $E(0) = 0$, so

$$E(t) \le 0 \tag{25}$$

Because $E(t) \ge 0$, $E(t) = 0$. Therefore, $w \equiv 0$, and $u_1 = u_2$. Then, we get the uniqueness for u_m by the same procedure of previous subsection for case $R(u) = -u^2$, we could extend this statement to the weak solution for (19), and uniqueness $L^2([0,T], H^1(\mathbb{T}))$ is obtained.

4.4. **Mild Solution.**

For the reaction-diffusion equation

$$\begin{cases} u_t - \nu u_{xx} = u - u^2 \ in \ [0,T]\times\mathbb{T} \\ u(t=0,x) = g(x) \end{cases} \tag{26}$$

We first do transformation $v = e^{-t}u$, so that one can convert (26) to the following:

$$\begin{cases} v_t - \nu v_{xx} = -e^t v^2 \ in \ [0,T]\times\mathbb{T} \\ v(t=0,x) = e^{-t}g(x) \end{cases} \tag{27}$$

Assume v is the mild solution of (27), then, $u = e^t v$ is the mild solution of (26). Now, focus on mild solution v.

Definition 4.1. *The solution satisfies:*

$$v(t,x) = S(t)g - \int_0^t [S(t-s)(v^2(s)e^s)]ds$$

is called a mild solution of (27). Where $S(t)g := \Phi(t,x) * g(x)$ *with* $\Phi(t,x)$ *denotes heat kernel.*

Now, let's focus on the existence and uniqueness of mild solution for equation (27), after comparing (27) and (13), notice that the only different part of proof between these two equations is the proof of assumption $||B[u,v]||_X \le c||u||_X||v||_X$

Notice that for any $t \in [0,T]$,

$$\begin{aligned} B[u,v] \quad &:= \quad \int_0^t \Phi(t-s) * [e^s u(s,\cdot)v(s,\cdot)]ds \\ &\le \quad e^T \int_0^t \Phi(t-s) * [u(s,\cdot)v(s,\cdot)]ds \end{aligned} \tag{28}$$

Hence, everything follows from the previous case where our reaction term was $R(u) = -u^2$, up to a constant.

5. **Reaction term** $R(u) = u(1-u)(\alpha - u)$

In this chapter, consider the following reaction-diffusion equation:

$$\begin{cases} u_t - \nu u_{xx} = u(1-u)(\alpha-u) \ in \ [0,T]\times\mathbb{T} \\ u(t=0,x) = g(x) \end{cases} \tag{29}$$

First, consider the following ode problem:

$$\begin{cases} u_t = u(1-u)(\alpha-u) \ in \ [0,T]\times\mathbb{T} \\ u(t=0,x) = b \ where \ 0 \le b \le 1 \end{cases} \tag{30}$$

One can easily check that $u = 0, 1, \alpha$ are three steady state solutions of this ODE. And $u = \alpha$ is the only stable solution. Then by the ODE theory, we know that if initial data b lies in $[0,1]$, our solution u will also lie in $[0,1]$. The diffusion term doesn't destroy this structure, hence, in this section, we assume the solution of (29) lies in $[0,1]$ with initial data provided in $[0,1]$.

5.1. **Existence.** The weak formulation is

$$< u_t, v > + \nu((u,v)) = < u(1-u)(\alpha-u), v > \ \forall v \in H^1(\mathbb{T});\ 0 < \alpha < 1 \tag{31}$$

5.1.1. *Uniform boundedness of* $\max_{0\le t\le T} ||u||^2_{L^2}$.

Choose $v = u_m \in V$ as test function in (31), assume $0 \le u \le 1$,

$$\int u_t u dx + \nu \int u_x^2 dx = \int u^2(1-u)(\alpha - u)dx$$

$$\frac{d}{dt}\int \frac{1}{2}u^2 dx + \nu \int u_x^2 dx = \alpha \int u^2 dx - (\alpha+1)\int u^3 dx + \int u^4 dx \le (\alpha+1)\int u^2 dx$$

In the last inequality, we used the assumption $u \in [0,1]$, then $u^4 \le u^2$.
Define $E(t) = \int u^2 dx$

$$\frac{d}{dt}\int u^2 dx \le 2(\alpha+1)\int u^2 dx$$

$$\Rightarrow E'(t) \le 2(\alpha+1)E(t)$$

By Grönwall inequality,

$$\Rightarrow E(t) \le e^{2(\alpha+1)t}E(0) = e^{2(\alpha+1)t}||g||^2_{L^2} \forall t \in [0,T]$$

Hence,

$$\int u^2 dx \le e^{2(\alpha+1)t}||g||^2_{L^2}$$

$$\Rightarrow \max_{0\le t\le t} ||u||^2_{L^2} \le e^{2(\alpha+1)T}||g||^2_{L^2}.$$

5.1.2. *Uniform boundedness of* u_m *in* $L^2([0,T];H^1(\mathbb{T}))$.

Choose $v = u_m \in V_m$ as test function in (31), assume $0 \le u \le 1$,

$$\int u_t u dx + \nu \int u_x^2 dx = \int u^2(1-u)(\alpha - u)dx$$

$$\frac{d}{dt}\int \frac{1}{2}u^2 dx + \nu \int u_x^2 dx = \alpha \int u^2 dx - (\alpha+1)\int u^3 dx + \int u^4 dx \le (\alpha+1)\int u^2 dx$$

Define $E(t) = \int u^2 dx$

$$\frac{d}{dt}\int u^2 dx + 2\nu \int u_x^2 dx + 2\nu \int u^2 dx \le 2(\alpha+1)\int u^2 dx + 2\nu \int u^2 dx$$

$$\Rightarrow E'(t) + 2\nu ||u||^2_{H^1} \le 2(\alpha+\nu+1)E(t)$$

By Grönwall inequality,

$$E(t) \le e^{\int_0^t 2(\alpha+\nu+1)ds}\left(E(0) + \int_0^t -2\nu||u(s,\cdot)||^2_{H^1} ds\right)$$

Solve for $||u||_{L^2H^1}$ we get

$$e^{2(\alpha+\nu+1)t} 2\nu ||u||_{L^2H^1} \le e^{2(\alpha+\nu+1)t}E(0) - E(t)$$

$$\Rightarrow ||u||_{L^2H^1} \le \frac{E(0)}{2\nu} = \frac{||g||^2_{L^2}}{2\nu}$$

5.1.3. *Uniform boundedness of u_{m_t} in $L^2([0,T];H^{-1}(\mathbb{T}))$.*

Let $v_1 \in V_m$, by the same decomposition of $H^1(\mathbb{T})$ as previously, we get:

$$\langle u_t, v\rangle = \alpha(u,v) - (\alpha+1)(u^2,v) + (u^3,v) - \nu((u,v)) \ \forall v \in H^1(\mathbb{T}).$$

Then,

$$\begin{aligned} |\langle u_t, v\rangle| &\leq \alpha|(u,v)| + (\alpha+1)|(u^2,v)| + |(u^3,v)| + \nu|((u,v))| \\ &\leq \alpha||u||_{H^1}||v||_{H^1} + (\alpha+1)||u||_{L^2}||v||_{L^2} + ||u||_{L^2}||v||_{L^2} + \nu||u||_{H^1}||v||_{H^1} \\ &\leq (2\alpha+2+\nu)||u||_{H^1}||v||_{H^1} \end{aligned}$$

$$\Rightarrow \frac{|\langle u_t, v\rangle|}{||v||_{H^1}} \leq (2\alpha+2+\nu)||u||_{H^1}$$

By the definition $||u_t||_{H^{-1}} = \sup_{v\in H^1(\mathbb{T})} \frac{|\langle u_t, v\rangle|}{||v||_{H^1}}$, then we have

$$||u_t(t,\cdot)||_{H^{-1}} \leq (2\alpha+2+\nu)||u||_{H^1}$$

Take L^2 integral over $[0,T]$

$$\int_0^T ||u_t(s,\cdot)||^2_{H^{-1}} ds \leq (2\alpha+2+\nu)^2 \int_0^T ||u(s,\cdot)||^2_{H^1} ds = (2\alpha+2+\nu)^2||u||_{L^2H^1}$$

Since u_{mt} is uniformly bounded in L^2H^{-1}, by Banach-Alaoglu theorem, u_{mt} converges weakly to a limit point u' in L^2H^{-1} up to a subsequence.

5.2. **Passing the limit to the nonlinear terms.**

$$(u(1-u)(\alpha-u), v) = \left(u^3,v\right) - (\alpha+1)\left(u^2,v\right) + \alpha(u,v)$$

By the uniform boundedness of $\max_{0\leq t\leq T}||u||^2_{L^2}$ and Banach-Alaoglu Theorem, we know that

$$(u_m, v) \to (u,v)$$

By the same procedure in 3.3.4 by Aubin-Lions Lemma, we know that

$$\left(u_m^2, v\right) \to \left(u^2, v\right)$$

Assume $u_m^3 - u^3 \geq 0$. Then

$$\begin{aligned} \int_{\mathbb{T}} \left(u_m^3 - u^3\right) v dx &= \int_{\mathbb{T}} (u_m - u)\, v \left(u_m^2 + u_m u + u^2\right) dx \\ &\leq 3\int_{\mathbb{T}} (u_m - u)\, v dx \end{aligned}$$

Note that $\max_{0\leq t\leq T}||u_m||^2_{L^2}$ is uniformly bounded. Then by the Banach-Alouglu theorem, there exists a subsequence u_{m_j} such that $u_{m_j} \rightharpoonup u$ in L^2L^2. Thus $3\int_{\mathbb{T}}\left(u_{m_j} - u\right) v dx \to 0$. Note that if $u_m^3 - u^3 \leq 0$, then repeating the proof with $\int_{\mathbb{T}}\left(u_m^3 - u^3\right) v dx$ works. Thus

$$\left(u_m^3, v\right) \to \left(u^3, v\right)$$

Thus,

$$\begin{aligned} \lim_{j\to\infty}\left(\left(u_{m_j t}, v\right) + \nu\left(\left(u_{m_j}, v\right)\right)\right) &= \lim_{j\to\infty}\left(u_{m_j}\left(1-u_{m_j}\right)\left(\alpha - u_{m_j}\right), v\right) \\ &= \lim_{j\to\infty}\left(\left(u_{m_j}^3, v\right) - (\alpha+1)\left(u_{m_j}^2, v\right) + \alpha\left(u_{m_j}, v\right)\right) \\ \langle u_t, v\rangle + \nu((u,v)) &= \left(u^3,v\right) - (\alpha+1)\left(u^2,v\right) + \alpha(u,v) \\ &= \langle u(1-u)(\alpha-u), v\rangle \end{aligned}$$

for $v \in H_0^1$. By the same procedure in 3.3.6, thus, u is a weak solution in $L^2\left([0,T], H^1(\mathbb{T})\right)$.

5.3. **Uniqueness.** Let $u_1, u_2 \in L^2\left([0,T], H^1(\mathbb{T})\right)$ be weak solutions of

$$\begin{cases} u_t - \nu u_{xx} = u(1-u)(\alpha - u) \\ u(t=0,x) = g(x) \end{cases} \tag{32}$$

Let $w = u_1 - u_2$. Then w satisfies

$$\begin{cases} w_t - \nu w_{xx} = \alpha w - (\alpha+1) w\left(u_1+u_2\right) + w\left(u_1^2 + u_1 u_2 + u_2^2\right) \\ w(t=0,x) = 0 \end{cases} \tag{33}$$

The weak formulation associated with (33) is

$$(w_t, v) + \nu((w,v)) = \alpha(w,v) - (\alpha+1)\left(w\left(u_1+u_2\right), v\right) + \left(w\left(u_1^2+u_1u_2+u_2^2\right), v\right) \tag{34}$$

for $v \in V_m$ and where $(f,g) = \int_{\mathbb{T}} fg dx$ and $((f,g)) = \int_{\mathbb{T}} f_x g_x dx$. Now choose $v = w$. Because $0 \le u_1 \le 1$ and $0 \le u_2 \le 1$, $u_1 + u_2 \le 2$ and $u_1^2 + u_1 u_2 + u_2^2 \le 3$. Define $E := \int_{\mathbb{T}} w^2 dx$. Then

$$\begin{aligned}
(w_t, w) + \nu((w,w)) &= \alpha(w,w) - (\alpha+1)\left(w\left(u_1+u_2\right), w\right) + \left(w\left(u_1^2+u_1u_2+u_2^2\right), w\right) \\
\int_{\mathbb{T}} w_t w dx + \nu \int_{\mathbb{T}} w_x^2 dx &= \alpha \int_{\mathbb{T}} w^2 dx - (\alpha+1)\int_{\mathbb{T}} w^2\left(u_1+u_2\right) dx + \int_{\mathbb{T}} w^2\left(u_1^2+u_1u_2+u_2^2\right) dx \\
\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}} w^2 dx &\le \alpha \int_{\mathbb{T}} w^2 dx + 2(\alpha+1)\int_{\mathbb{T}} w^2 dx + 3\int_{\mathbb{T}} w^2 dx \\
\frac{1}{2}E'(t) &\le (\alpha + 2(\alpha+1) + 3)E(t) \\
E'(t) &\le (6\alpha + 10)E(t)
\end{aligned}$$

Then by Grönwall's inequality,

$$E(t) \le E(0) e^{\int_0^T (6\alpha+10)dt} \tag{35}$$

Because $E(0) = 0$ and $\int_0^T (6\alpha+10)dt$ is finite, $E(t) \le 0$. Thus, $u_1 \equiv u_2$, i.e. the solution of (32) is unique.

5.4. **Mild solution.**

The proof of $R(u) = u(1-u)$ could survive successfully here up to the constant 2 by the fact that $0 \le u \le 1$, $|\alpha - u| \le 2$.

6. Regularity of weak solution

In this section, we show the instant regularization of reaction equations discussed in this paper. Consider

$$u_t - \nu u_{xx} = -u^2$$

6.1. H^k **energy estimate.**

$$\begin{aligned}
\int \partial_x^k u \cdot \partial_x^k \left(u_t - \nu u_{xx} + u^2\right) dx &= 0 \qquad (36) \\
\frac{d}{dt}\frac{1}{2}\int |\partial_x^k u|^2 dx - \nu \int \partial_x^k u \cdot \partial_x^{k+2} u dx + \int \partial_x^k u \cdot \partial_x^k (u^2) dx &= 0
\end{aligned}$$

Denote $I_1 = -\nu \int \partial_x^k u \cdot \partial_x^{k+2} u dx$, $I_2 = \int \partial_x^k u \cdot \partial_x^k (u^2) dx$. For I_1, by integration by parts,

$$I_1 = \nu \int |\partial_x^{k+1} u|^2 dx \ge 0$$

By Leibniz rule $\partial_x^k (u^2) = \sum_{j=0}^k \binom{k}{j} \partial_x^k u \cdot \partial_x^{k-j} u$, we deal with the j^{th} term of I_2, apply Hölder inequality, we get

$$\int \partial_x^k u \cdot \partial_x^j \cdot \partial_x^{k-j} u dx \le ||\partial_x^k u||_{L^2} ||\partial_x^j u||_{L^{p_1}} ||\partial_x^{k-j} u||_{L^{p_2}}$$

where $\dfrac{1}{p_1} + \dfrac{1}{p_2} = \dfrac{1}{2}$

By Gragliardo-Nirenberg interpolation equation, we know

$$\begin{aligned}
||\partial_x^j u||_{L^{p_1}} &\le C_1 ||\partial_x^k u||_{L^{\alpha_1}} ||u||_{L^\infty}^{1-\alpha_1} \ \textit{with} \ \frac{1}{p_1} = j + (\frac{1}{2} - k)\alpha_1 + \frac{1-\alpha_1}{\infty} \qquad (37) \\
||\partial_x^{k-j} u||_{L^{p_2}} &\le C_2 ||\partial_x^k u||_{L^{\alpha_1}} ||u||_{L^\infty}^{1-\alpha_1} \ \textit{with} \ \frac{1}{p_1} = j + (\frac{1}{2} - k)\alpha_1 + \frac{1-\alpha_1}{\infty}
\end{aligned}$$

Notice

$$\alpha_1+\alpha_2=\frac{\frac{1}{p_1}-j}{\frac{1}{2}-k}+\frac{\frac{1}{p_2}-k+j}{\frac{1}{2}-k}=\frac{\frac{1}{p_1}+\frac{1}{p_2}-k}{\frac{1}{2}-k}=1$$

we get $|I_2|\le C||\partial_x^k u||_{L^2}^2||u||_{L^\infty}$ where C is a constant depends on k and domain $\mathbb{T}$.
Denote $E_k(t):=\int|\partial_x^k u|^2dx$,

we get

$$\begin{aligned}\frac{1}{2}E_k'(t)+I_1 &\le CE_k(t)||u||_{L^\infty}\\ \textit{by maximum principle}, &\le CE_k(t)||g||_{L^\infty}\end{aligned} \tag{38}$$

Note $I_1\ge 0$, hence, $E_k'(t)\le 2CE_k(t)||g||_{L^\infty}$, by Grönwall inequality, we get H^k energy estimate

$$E_k(t)\le||\partial_x^k g||_{L^2}^2e^{2CT||g||_{L^\infty}}$$

Claim 6.1. *$\forall m\in N, t^m||\partial_x^{k+m}u(t,\cdot)||_{L^2}^2$ is uniformly bounded in t, namely, $\forall t\in[0,T]$, $||\partial_x^{k+m}u(t,\cdot)||_{L^2}^2\le C(m,T)t^{-m}$, where $C(m,T)$ is a constant depends on m and T.*

Proof. We do induction on m.

(1) $m=0$, the argument is true by previous H^k energy estimate, with $C(0,T)=||\partial_x^k g||_{L^2}^2e^{CT||g||_{L^\infty}}$.
(2) Suppose the argument is true for m.
(3) Show it is true for $m+1$.

Consider the H^s energy estimate for $(k+m)th$ order derivative, we get the similar inequality by replace k with $k+m$. Denote $E_{k+m}(t):=\int|\partial_x^{k+m}u|^2dx$,

$$\frac{1}{2}E_{k+m}'(t)+\nu\int|\partial_x^{k+m+1}u|^2dx\le CE_{k+m}(t)||g||_{L^\infty} \tag{39}$$

Note $\int|\partial_x^{k+m+1}u|^2dx\ge 0$, then we get

$$\frac{1}{2}E_{k+m}'(t)\le CE_{k+m}(t)||g||_{L^\infty}$$

By Grönwall inequality over $[\frac{t}{2},t]$, for any $\tau\in[\frac{t}{2},t]$,

$$\begin{aligned}E_{k+m}(\tau) &\le E_{k+m}(\frac{t}{2})e^{\int_{\frac{t}{2}}^t 2C||g||_{L^\infty}ds}\\ \textit{By case}(2) &\le C(m,T)(\frac{t}{2})^{-m}e^{CT||g||_{L^\infty}}\end{aligned} \tag{40}$$

Integrate (39) over $[\frac{t}{2},t]$, we get

$$\begin{aligned}\int_{\frac{t}{2}}^t||\partial_x^{k+m+1}u(s,\cdot)||_{L^2}^2ds &\le \frac{1}{2\nu}\left(\int_{\frac{t}{2}}^t 2C||g||_{L^\infty}E_{m+k}(s)ds-E_{m+k}(t)+E_{m+k}(\frac{t}{2})\right)\\ &\le \frac{1}{2\nu}\left(\int_{\frac{t}{2}}^t 2C||g||_{L^\infty}E_{m+k}(s)ds+E_{m+k}(\frac{t}{2})\right)\\ by(40) &\le \frac{1}{2\nu}\left(2TC||g||_{L^\infty}+1\right)\left(\frac{t}{2}\right)^{-m}C(m,T)+e^{CT||g||_{L^\infty}}\\ &\le C'(m,T)\left(\frac{t}{2}\right)^{-m}\end{aligned}$$

By mean value theorem, we know

$$\int_{\frac{t}{2}}^t||\partial_x^{k+m+1}u(s,\cdot)||_{L^2}^2ds=\frac{t}{2}||\partial_x^{k+m+1}u(\tau_1,\cdot)||_{L^2}^2\ \textit{for some}\ \tau\in[\frac{t}{2},t]$$

Hence,

$$E_{m+k+1}(\tau)=||\partial_x^{k+m+1}u(\tau,\cdot)||_{L^2}^2\le C(m+1,T)t^{-m-1}$$

Finally, we run H^{k+m+1} energy estimate from τ to t, we get

$$||\partial_x^{k+m+1}u(t,\cdot)||_{L^2}^2\le\frac{1}{2\gamma}\left(\int_\tau^t 2CE_{k+m+1}(s)||g||_{L^\infty}ds-E_{k+m+1}(t)+E_{k+m+1}(\tau)\right)\le C(m+1,T)t^{-m-1}$$

The result follows. □

6.2. **Extend result to other cases.**

(1) For reaction term $R(u) = u(1-u) = u - u^2$
The only extra term is u, when we do H^s energy estimate, this term u only gives us $-\int \partial_x^k u * \partial_x^k u dx = -E_k(t)$ which cause no trouble at all.

(2) For reaction term $R(u) = u(1-u)(\alpha - u)$
Since $0 \leq u \leq 1$, the extra term u^3 could be bounded by u^2 or u which go back to the the case $R(u) = -u^2$ or $R(u) = u(1-u)$.

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Partial Differential Equation, Department of Mathematics, University of South Carolina
Email address: chunyan@email.sc.edu