SPHERICAL HARMONICS

CHUNYAN LI

Abstract. In this report, we develop the basic theory of spherical harmonics including orthogonality, Legendre polynomials, the decomposition of $L^2(S^{d-1})$ and the applications. We start from the homogeneous polynomials which is the foundation of spherical harmonics to develop the orthogonality of spherical harmonics. Then we develop the relationship between spherical harmonics and Legendre polynomials. Meanwhile, we conclude that the kernel $F(\xi, \eta)$ can be represented in terms of the Legendre polynomials which is very useful for explicit expansion of $f \in L^2(S^{d-1})$. When we focus on the applications of spherical harmonics on the unit ball, we show that all spherical harmonics form a complete basis of the Hilbert space $L^2(S^{d-1})$. That is we could decompose $L^2(S^{d-1})$ into the direct sum of all $\mathscr{H}_n$. We also discuss the the application of spherical harmonics on the Dirichlet problem for the Laplace equation on the ball based on the decomposition of $L^2(S^{d-1})$.

1. Introduction

The harmonic functions are solutions of the Laplace equation. They play a fundamental role in Analysis and other areas of mathematics and in applications. The spherical harmonics are the traces of harmonic polynomials on the sphere and are the analogue of the trigonometric system on the system. They play an equally important role. In this paper, we develop the basic theory of spherical harmonics including orthogonality, Legendre polynomials and the decomposition of $L^2(S^{d-1})$. We first introduce important definitions and theorems of Hilbert space which will be used in the last section. In the following section, we focus on the polynomial solution of Laplace equation in 2 and 3 dimensions which will motivate discussion of the spherical harmonics in high dimensions. Then we are ready to develop the main theory of spherical harmonics. We show that all spherical harmonics form a complete basis of Hilbert space $L^2(S^{d-1})$. Then we focus on the expansion of $f \in L^2(S^{d-1})$ which based on the decomposition of $L^2(S^{d-1})$. For the convenience of application, we expansion f in terms of kernel $F(\xi, \eta)$ instead of the explicit expression of spherical harmonics. Much of the theory developed of this paper references Hochstadt [1]. To find the development of the spherical harmonics that arise in R^3, one can look in almost any text on mathematical methods, electrodynamics, or quantum mechanics, physical geodesy (see [3], [4]) The classical work by Müller [2] is also recommended for the readers who are eager for extensive content on spherical harmonics.

2. Background

In this chapter, we will begin with important definitions and Theorems which will be used later.

2.1. Hilbert Space.

Definition 2.1. (1) *In a normed space, a sequence $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence if $\forall \epsilon > 0$, $\exists$ an integer $N > 0$ such that $\|x_n - x_m\| < \epsilon$ for all $n, m \geq N$.*

(2) *A normed space is complete if every Cauchy sequence converges to an element in the space.*

(3) *A complete inner product space is a Hilbert space.*

(4) *An orthonormal set $\{\phi_n\}_{n=0}^{\infty} \subset H$ is complete if $\forall f \in H$, there exist scalars $c_1, c_2, ...$ such that*

$$\lim_{n\to\infty} \|f - \sum_{k=0}^{n} c_k \phi_k\| = 0. \tag{1}$$

Theorem 2.2. *An orthonormal set $\{\phi_n\}_{n=0}^{\infty} \subset H$ is complete if and only if Parseval's equality $\|f\|^2 = \sum |(f, \phi_n)|^2$ holds for each $f \in H$.*

Definition 2.3. *A set $\{\phi_n\}_{n=0}^{\infty}$ is closed if $(f, \phi_n) = 0$ $\forall n$ implies $f = 0$.*

Theorem 2.4. *The orthonormal set $\{\phi_n\}_{n=0}^{\infty} \subset H$ is complete if and only if it is closed.*

Proof. (Sufficient) Let $\{\phi_n\}_{n=0}^{\infty} \subset H$ be a closed orthonormal set and $f \in H$. We need to show that

$$\lim_{n\to\infty} \left(\|f\|^2 - \sum_{k=0}^{n} (f, \phi_k)^2 \right) = 0$$

Date: 2020 Spring.

The author would like to thank P. Petrushev for his patient explanations of both the mathematics and latex coding necessary to create a term paper of this level.

by theorem 2.2.
Define $g_n = f - \sum_{k=0}^{n}(f,\phi_k)\phi_k$, first show that $\{g_n\}_{n=0}^{\infty}$ is a Cauchy sequence. If $n > m$,

$$\begin{aligned} \|g_n - g_m\|^2 &= \|\sum_{k=m+1}^{n}(f,\phi_k)\phi_k\|^2 \\ &= \sum_{k,l=m+1}(f,\phi_k)(f,\phi_l)(\phi_k,\phi_l) \\ &= \sum_{k=m+1}^{n}(f,\phi_k)^2 \end{aligned}$$

since the series $\sum_{k=0}^{\infty}(f,\phi_k)$ converges, $\sum_{k=m+1}^{n}(f,\phi_k)^2$ can be arbitrarily small for large enough n, m, then, $\{g_n\}_{n=0}^{\infty}$ is a Cauchy sequence. Hence, there exist $g \in H$ such that

$$\lim_{n\to\infty}\|g_n - g\| = 0 \tag{2}$$

by the completeness of H. Now fix j and taken $n \in N$ such that for any $n > j$, By the Cauchy-Schwartz inequality

$$|(g,\phi_j)| = |(g_n - g,\phi_j)| \leq \|g_n - g\|\|\phi_j\| = \|g_n - g\|.$$

Hence, we get

$$|(g,\phi_j) \leq \lim_{n\to\infty}\|g_n - g\| = 0$$

which implies that for any j, $(g,\phi_j) = 0$. Since $\{\phi_n\}_{n=0}^{\infty}$ is closed, then, we have $g = 0$. Therefore,

$$\lim_{n\to\infty}\left(\|f\|^2 - \sum_{k=0}^{n}(f,\phi_k)^2\right) = \lim_{n\to\infty}\|g_n\|^2 = \lim_{n\to\infty}\|g_n - g\|^2 = 0.$$

(Necessary) Suppose $\{\phi_n\}_{n=0}^{\infty}$ is complete but not closed. Then there exists a nonzero function f such that $(f,\phi_n) = 0$ for every n. Then,

$$\lim_{n\to\infty}\left(\|f\|^2 - \sum_{k=0}^{n}(f,\phi_k)^2\right) = \|f\|^2 \neq 0$$

Hence, by 2.2, $\{\phi_n\}_{n=0}^{\infty}$ is not complete which is a contradiction.

2.2. **surface area of unit sphere in d dimension.** We shall find the surface area of a unit sphere in d dimension which will be used later. Let $f(r)$ be any function of $r = \sqrt{x_1^2 + ... + x_d^2}$ for which the integral

$$I_d = \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} f(r)dx_1dx_2...dx_d \tag{3}$$

exists.
To perform the above integration we shall make use of the spherical symmetry of the function $f(r)$. Let ω_d denote the surface area of a unit sphere. Then we can evaluate I_d by integrating over spherical shells, so that

$$\begin{aligned} I_d &= \int_0^{\infty}\int_{S(r)} f(r)d\omega dr \\ &= \int_0^{\infty} f(r)\omega_d(r)dr \\ &= w_d\int_0^{\infty} f(r)r^{d-1}dr \end{aligned} \tag{4}$$

since the surface area of a sphere or radius r is $\omega_d r^{d-1}$. It follows that

$$\omega_d = \frac{\int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} f(r)dx_1...dx_d}{\int_0^{\infty} f(r)r^{d-1}dr}$$

Since w_{d-1} is independent with the choice of $f(r)$, in particular, we could choose $f(r) = e^{-r^2} = e^{-(x_1^2+...+x_d^2)}$ we get

$$\omega_d = \frac{[\int_{-\infty}^{\infty} e^{-x^2}dx]^d}{\int_0^{\infty} e^{-r^2}r^{d-1}dr} = \frac{[\sqrt{\pi}]^d}{\frac{1}{2}\Gamma(d/2)} = \frac{2\pi^{d/2}}{\Gamma(d/2)}.$$

3. Spherical Harmonic in 3−dimension

In this chapter, we will find the solutions of Laplace equation in 2 and 3 dimensions to motivate the set up of spherical harmonics.

3.1. **2-D Laplace equation.** The Laplace operator in 2−dimension is given by

$$\Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

The relationship between Cartesian coordinate and polar coordinate is given by

$$x = rcos\phi, \qquad y = rsin\phi.$$

Now we derive the Laplace operator in polar coordinate.

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta}$$

Finally, we get the Laplace operator in polar coordinate as follows:

$$\Delta_2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} \tag{5}$$

Thus, we get the Laplace equation

$$\Delta_2 \Phi = \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \phi^2} = 0$$

We assume the solution of this equation has the form $\Phi(r, \phi) = X(r)Y(\phi)$. Inserting this solution to the above equation we get:

$$Y\frac{\partial^2 X}{\partial r^2} + Y\frac{1}{r}\frac{\partial X}{\partial r} + X\frac{1}{r^2}\frac{\partial^2 Y}{\partial \phi^2} = 0$$

Multiplying by r^2/XY and rearranging,

$$\frac{r^2}{X}\frac{\partial^2 X}{\partial r^2} + \frac{r}{X}\frac{\partial X}{\partial r} = -\frac{1}{Y}\frac{\partial^2 Y}{\partial \phi^2}$$

Observe that the left hand side is the function of r alone and the right hand side is the function of ϕ alone, in order to make the above equality holds, both sides of the equation are the same constant. Denote $-\lambda$, we get:

$$\frac{1}{Y}\frac{\partial^2 Y}{\partial \phi^2} = \lambda$$

$$\frac{r^2}{X}\frac{\partial^2 X}{\partial r^2} + \frac{r}{X}\frac{\partial X}{\partial r} = -\lambda$$

The linearly independent solutions of the second order ODE $Y^{''} = \lambda Y$ are

$$Y(\phi) = \begin{cases} e^{\sqrt{\lambda}\phi}, e^{-\sqrt{\lambda}\phi} & \text{if } \lambda > 0, \\ 1, \phi & \text{if } \lambda = 0, \\ \sin(\sqrt{|\lambda|\phi}), \cos(\sqrt{|\lambda|\phi}) & \text{if } \lambda < 0 \end{cases}$$

Since (r_0, ϕ_0) represents the same point as $(r_0, \phi_0 + 2k\pi)$ for any $k \in Z$, $Y(\phi)$ is a periodic function with period 2π in this problem. Thus, the linearly independent solutions of this problem are

$$1, \sin(\sqrt{\lambda}\phi), \cos(\sqrt{\lambda}\phi)$$

where λ is a non-negative integers. Then, we could replace λ with $-m^2$ then, the solutions are

$$1, Y_{1,n} = \cos(n\phi), Y_{2,m} = \sin(m\phi)$$

3.2. **3-D Laplace equation.** The Laplace operator in $3 - dimension$ is given by

$$\Delta_3 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Let us first derive the Laplace operator in spherical coordinates.
Since

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta.$$

Using chain rule,we get

$$\Delta_3 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial \phi^2}\right] \tag{6}$$

We could derive the spherical harmonics in $3 - D$ through solving Laplace equation $\Delta_3 \Phi = 0$ in spherical coordinates via separating variables.

Assume the solution has the form $\Phi(r,\theta,\phi) = X(r)Y(\theta,\phi)$. Then, we insert this solution into the Laplace equation (6), we get

$$\Delta_3 = Y(\theta,\phi)\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial X(r)}{\partial r}\right) + X(r)\frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y(\theta,\phi)}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y(\theta,\phi)}{\partial\phi^2}\right] = 0 \tag{7}$$

Multiplying by $\frac{r^2}{X(r)Y(\theta,\phi)}$ and rearranging,

$$\frac{1}{X}\frac{\partial}{\partial r}\left(r^2\frac{\partial X(r)}{\partial r}\right) = -\frac{1}{Y}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y(\theta,\phi)}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y(\theta,\phi)}{\partial\phi^2}\right]$$

Notice that the right hand side is a function of r and the left hand side is a function of θ and ϕ, in order to let this equality holds, both sides of the above equation must be the same constant, denote $-\lambda$.It follows that

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y(\theta,\phi)}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y(\theta,\phi)}{\partial\phi^2}\right] = \lambda Y(\theta,\phi) \tag{8}$$

It turns out that the $Y's$ which satisfy this equation are actually spherical harmonics. Observe the above equation, we see that the $Y's$ are eigenfunctions of the angular part of the Laplace operator. In the following sections we will prove that the functions $Y's$ form a complete set over the unit sphere, each Y is a homogeneous polynomial restricted to the unit sphere, and these polynomials satisfy the Laplace equation.

4. Spherical Harmonic in $d-dimension$

Definition 4.1. *A polynomial $H_n(x_1,x_2,...,x_d)$ is homogeneous of degree n in d variables $x_1,x_2,...,x_d$ if*

$$H_n(tx_1,tx_2,...,tx_d) = t^n H_n(x_1,...,x_d) \tag{9}$$

Definition 4.2. *A polynomial H_n of degree n in d variables $x_1,x_2,...,x_d$ is harmonic if $\Delta H_n = 0$, where $\Delta = \sum_{i=1}^{d}\frac{\partial^2}{\partial x_i^2}$*

Definition 4.3. *A spherical harmonic of degree n, denoted $S_n(\xi)$, is a harmonic homogeneous polynomial of degree n in d variables restricted to the unit $(d-1)-Sphere$. That is, $S_n = H_n|_{S^{d-1}}$, $S_n : S^{d-1} \to R$, given by $S_n(\xi) = H_n(\xi)$ for every $\xi \in S^{d-1}$ for some harmonic homogeneous polynomial H_n.*

4.1. **The number of spherical harmonics.** We first find the number of linearly independent homogeneous polynomials.

Lemma 4.4. *If $K(d,n)$ denotes the number of linearly independent homogeneous polynomials of degree n in d variables, then*

$$K(d,n) = \frac{(d+n-1)!}{n!(d-1)!} \tag{10}$$

Proof. We can expand $H_n(x_1,x_2,...,x_d)$ as a polynomial in variable x_d, so that

$$H_n(x_1,x_2,...,x_d) = \sum_{j=0}^{n} x_d^j A_{n-j}(x_1,x_2,...,x_{d-1}) \tag{11}$$

where $A_{n-j}(x_1,x_2,...,x_{d-1})$ denotes the corresponding homogeneous polynomial of degree $n-j$ in $d-1$ variables. By the assumption, we know for each A_{n-j} there are $K(d-1,n-j)$ linearly independent choices so that $K(d,n)$ satisfies

$$K(d,n) = \sum_{j=0}^{n} K(d-1,n-j) = \sum_{j=0}^{n} K(d-1,j) \tag{12}$$

We shall solve it by the method of generating functions. Let

$$M(d) := \sum_{n=0}^{\infty} r^n K(d,n) \tag{13}$$

If we substitute from (12) in (13) and interchange the n and j summations, we obtain

$$\begin{aligned} M(d) &= \sum_{n=0}^{\infty}\sum_{j=0}^{n} r^n K(d-1,j) \\ &= \sum_{j=0}^{\infty} K(d-1,j) \sum_{n=j}^{\infty} r^n \\ &= \frac{1}{1-r}\sum_{j=0}^{\infty} r^j K(d-1,j) \\ &= \frac{M(d-1)}{1-r} \end{aligned} \tag{14}$$

From the recursion formula above, we deduce that

$$M(d) = \frac{M(1)}{(1-r)^{d-1}}$$

For $d = 1$, clearly, $K(1, n) = 1$, since $H_n(x_1) = cx_1^n$, so that $M(1) = \frac{1}{1-r}$ and

$$M(d) = (1-r)^{-d} = \sum_{n=0}^{\infty} \frac{(d+n-1)!}{n!(p-1)!} r^n$$

Comparing these two expressions of $M(d)$, since r, r^2,...,r^n,... are linearly independent, we obtain

$$K(d,n) = \frac{(d+n-1)!}{n!(d-1)!}$$

Now, Let's find the number of linearly independent homogeneous harmonic polynomials which is the number of linearly independent spherical harmonics.

Theorem 4.5. *If $N(d, n)$ denotes the number of linearly independent homogeneous harmonic polynomials of degree n in d variables, then*

$$N(d,n) = \frac{2n+d-2}{n}\binom{n+d-3}{n-1}$$

Proof. We can decompose the operator Δ_d into two operators.

$$\Delta_d = \frac{\partial^2}{\partial x_d^2} + \Delta_{d-1}$$

where Δ_{d-1} denotes the $d-1$ dimensional Laplace operator acting on functions of x_1, x_2, ...,x_{d-1}. Then,

$$\begin{aligned} \Delta_d H_n &= \Delta_d\Big(\sum_{j=0}^{n} x_d^j A_{n-j}(x_1.x_2,...,x_d)\Big) \\ &= \sum_{j=2}^{n} j(j-1)x_d^{j-2}A_{n-j} + \sum_{j=0}^{n} x_d^j \Delta_{d-1}A_{n-j} \\ &= \sum_{j=0}^{n} x_d^j[(j+1)(j+2)A_{n-j-2} + \Delta_{d-1}A_{n-j}] \\ &= 0 \end{aligned} \tag{15}$$

In the above $A_{-1} = A_{-2} = 0$. Since 1,x_d, x_d^2,...,x_d^n are linearly independent, we require that

$$\begin{aligned} \Delta_{d-1}A_n + 2A_{n-2} &= 0 \\ \Delta_{d-1}A_{n-1} + 6A_{n-3} &= 0 \\ \Delta_{d-1}A_{n-2} + 12A_{n-4} &= 0 \\ &\dots \\ \Delta_{d-1}A_2 + n(n-1)A_0 &= 0 \\ \Delta_{d-1}A_1 &= 0 \\ \Delta_{d-1}A_0 &= 0 \end{aligned}$$

From the above we see that once A_n and A_{n-1} are selected all remaining A_{n-j} are determined recursively. From 4.4 we know that A_n can be written as the linear combination of $K(d-1, n)$ polynomials, hence, we need

$K(d-1,n)$ coefficients to determine A_n, Similarly, we need $K(d-1,n-1)$ coefficients to determine A_{n-1}. Hence, we can select A_n and A_{n-1} in $K(d-1,n)+K(d-1,n-1)$ ways so that

$$\begin{aligned} N(d,n) &= K(d-1,n)+K(d-1,n-1) \\ &= \binom{n+d-2}{n}+\binom{n+d-3}{n-1} \\ &= \frac{n+d-2}{n}\binom{n+d-3}{n-1}+\binom{n+d-3}{n-1} \\ &= \frac{2n+d-2}{n}\binom{n+d-3}{n-1} \end{aligned} \tag{16}$$

For example, for $d=3$, we find that

$$N(3,n)=2n+1$$

Using spherical coordinates, these $(2n+1)$ functions are given by

$$r^n P_n^m(cos\theta)e^{im\phi}, m=-n,-n+1,...,n-1,n$$

In general, we write

$$H_n(X)=H_n(r\xi)=r^n S_n(\xi).$$

where $X=(x_1,x_2,...x_d)$ and ξ is the unit vector $\xi=(\xi_1,\xi_2,...,\xi_d)$, we refer to $S_n(\xi)$ as a spherical harmonic.

Theorem 4.6. (Orthogonality of Spherical Harmonics) *Let $S_n(\xi)$,$S_m(\xi)$ be two spherical harmonics, then,*

$$\int_{S^{d-1}} S_n(\xi)S_m(\xi)d\omega_d=0 \ if \ n\neq m \tag{17}$$

That is, spherical harmonics of different degrees are orthogonal over the sphere.

Proof. Let $H_n(X)$ denote the homogeneous polynomial of degree n corresponding to $S_n(\xi)$. Then,

$$H_n(tX)=t^n H_n(X)$$

Differentiation of the above with respect to t yields

$$\frac{d}{dt}H_n(tX)=nt^{n-1}H_n(X) \tag{18}$$

But

$$\frac{d}{dt}H_n(tX)=\sum_{i=1}^{d}\frac{\partial H_n(tX)}{\partial x_i}\frac{dtx_i}{dt} \tag{19}$$

In particular for $t=1$ we find

$$\sum\frac{\partial H_n(X)}{\partial x_i}x_i=nH_n(X) \tag{20}$$

Now let X be such that $|X|=1$ so that $X=\xi$, and the above equation reduces to

$$\sum_{i=1}^{d}\frac{\partial S_n(\xi)}{\partial xi_i}\xi_i=nS_n(\xi) \tag{21}$$

Let ν denote the exterior normal to the unit sphere $\xi=1$. It follows that

$$\frac{\partial S_n(\xi)}{\partial\nu}=\xi\nabla S_n(\xi)=\sum_{i=1}^{d}\frac{\nabla S_n(\xi)}{\partial\xi_i}\xi_i=nS_n(\xi) \tag{22}$$

Using divergence theorem in d dimensions we have

$$\begin{aligned} 0 &= \int_{|X|\leq 1}[H_n(X)\Delta_d H_k(X)-H_k(X)\Delta_d H_n(X)]dx_1dx_2...dx_d \\ &= \int_{|\xi|=1}[S_n(\xi)\frac{\partial S_k(\xi)}{\partial\nu}-S_k(\xi)\frac{\partial S_n(\xi)}{\nu}]d\omega_d \\ &= (k-n)\int_{|\xi|=1}S_n(\xi)S_k(\xi)d\omega_d \end{aligned} \tag{23}$$

and for $n\neq k$ the result follows.

By the theorem above 4.5 and 4.6, we could conclude that $\{S_{n,j}(\xi)\}_{j=1}^{N(d,n)}{}_{n=0}^{\infty}$ forms an orthonormal set. Suppose $\{S_n(\xi)\}$ is a set of $N(d,n)$ linearly independent real spherical harmonics of degree n. By the Gram-Schmidt process we can construct an orthonormal set

$$S_{n,1}(\xi), S_{n,2}(\xi), ..., S_{n,N(d,n)}(\xi).$$

By 4.6, we know all spherical harmonics with different degrees are orthogonal over the sphere. Combine these two results, we get that $\{S_{n,j}(\xi)\}_{j=1}^{N(d,n)}{}_{n=0}^{\infty}$ forms an orthonormal set. In the following, we will prove that $\{S_{n,j}(\xi)\}_{j=1}^{N(d,n)}{}_{n=0}^{\infty}$ is closed. But, before that, we first show that under the coordinate rotation, spherical harmonic of degree n is still a spherical harmonic. Suppose $\{S_n(\xi)\}$ is a set of $N(d,n)$ orthonormal real spherical harmonics of degree n. Then we have,

$$\int_{|\xi|=1} S_{n,j}(\xi)S_{n,k}(\xi)d\omega_d = \delta_{j,k} \tag{24}$$

Let A denote an orthogonal matrix that represents a rotation of the coordinate system. Since the integration is taken over the entire sphere, the orthonormal set of spherical harmonics remains orthonormal in a rotated coordinate frame. It follows that

$$\int_{|\xi|=1} S_{n,j}(A\xi)S_{n,k}(A\xi)d\omega_d = \delta_{j,k} \tag{25}$$

for all orthogonal matrices($A^TA = I$).

Proposition 4.7. *If $S_n(\xi)$ is a spherical harmonic of degree n, then $S_n^{'}(\xi) = S_n(A\xi)$ is also a spherical harmonic of degree n, for any rotation matrix A.*

Proof. Let $S_n(\xi)$ be a spherical harmonic o degree n. Then there exists a harmonic homogeneous polynomial $H_n(X)$ of degree n such that $S_n = H_n|_{S^{d-1}}$. Denote $H_n^{'}(X) = H_n(AX)$. We claim that $S_n^{'} = H_n^{'}|_{S^{d-1}}$. Too see this, first notice that $H_n^{'}(X)$ is a polynomial in $x_1, ..., x_d$. Indeed, $H_n^{'}(X) = H_n(AX)$ is a linear combination of powers of the $\sum_{j=1}^{d} A_{ij}x_j$ and thus a linear combination of powers of the x_j. Then, notice that $H_n^{'}$ is homogeneous of degree n,

$$H_n^{'}(tX) = H_n(tAX) = t^n H_n(AX) = t^n H_n^{'}(X)$$

Finally, notice that $H_n^{'}$ is harmonic.Restricting $H_n^{'}$ to the unit sphere thus gives a spherical harmonic of degree n, that is $S_n^{'}(\xi)$.

Remark 4.8. *If let $\mathscr{H}_n$ denote the vector space of all spherical harmonics of degree n in d dimension. The above result says that $\mathscr{H}_n$ is invariant under the coordinate rotation.*

We now show the function $F(\xi,\eta)$ which is known as the kernel of the orthogonal projector

$$F(\xi,\eta) = \sum_{j=1}^{N(d,n)} S_{n,j}(\xi)S_{n,j}(\eta) \tag{26}$$

is invariant under the coordinate rotation. As we shall see later this function has similar properties and plays a similar role as does $K_n(x,y)$, defined by the Christoffel-Darboux formula in the theory of orthogonal polynomials. Since the set $\{S_{n,j}(\xi)\}_{j=1}^{N(d,n)}$, is a maximal linearly independent set of spherical harmonics of degree n, it forms a basis for all such functions. From the above proposition, we know $S_{n,j}(A\xi)$ is a spherical harmonic of degree n, It follows that

$$S_{n,j}(A\xi) = \sum_{l=1}^{N(d,n)} C_{l,j}S_{n,l}(\xi) \tag{27}$$

By inserting (27) in (25) and using (24) we find that

$$\sum_{l=1}^{N(d,n)} C_{l,j}C_{n,l} = \delta_{j,k} \tag{28}$$

(28) expresses the fact that the matrix $C = (c_{i,j})$ defined in (27) is also orthogonal.
Using (27) and (28) we have

$$
\begin{aligned}
F(A\xi, A\eta) &= \sum_{j=1}^{N(d,n)} \left(\sum_{l=1}^{N(d,n)} C_{t,j} S_{n,l}(\xi) \right) \left(\sum_{m=1}^{N(d,n)} C_{m,j} S_{n,m}(\eta) \right) \\
&= \sum_{j=1}^{N(d,n)} S_{n,j}(\xi) S_{n,j}(\eta) \\
&= F(\xi, \eta)
\end{aligned} \tag{29}
$$

This result expresses the fact that $F(\xi, \eta)$ is invariant under rotations of the coordinate system. By using this property, we will prove that $F(\xi, \eta)$ is a function of the inner product of ξ and η. Let's denote the inner product between two vectors by

$$(\xi, \eta) = \sum_{i=1}^{d} \xi_i \eta_i$$

and clearly for a rotation A, the inner product is invariant,

$$(A\xi, A\eta) = (\xi, A^T A\eta) = (\xi, \eta)$$

where A^T is the transpose of A. For a rotation $A^T A = I$

Lemma 4.9. *The function $F(\xi, \eta)$ defined by $F(\xi, \eta) = \sum_{j=1}^{N(d,n)} S_{n,j}(\xi) S_{n,j}(\eta)$ is a function of (ξ, η) only. That is*

$$F(\xi, \eta) = \phi((\xi, \eta))$$

Proof. From the above, we know the inner product is invariant under coordinate rotation and $F(\xi, \eta)$ is invariant, too. We shall select our coordinate system in a special way. This can be done without loss of generality by a suitable rotation. There is a rotation A_1 such that

$$A_1\eta = \eta_1, \ A_1\xi = \xi_1$$

where

$$\eta_1 = (1, 0, 0, ..., 0), \ \xi_1 = (t, \sqrt{1 - t^2}, 0, ..., 0)$$

so that $(\xi_1, \eta_1) = t$. Since $F(\xi, \eta)$ is invariant under rotations, $F(\xi, \eta) = F(\xi_1, \eta_1,)$. Hence, $F(\xi, \eta)$ will be a polynomial in the two variables t and $\sqrt{1 - t^2}$, that is

$$F(\xi, \eta) = P(t, \sqrt{1 - t^2})$$

But there is certainly a rotation A such that

$$A\eta_1 = \eta_1, \ A\xi_1 = (t, -\sqrt{1 - t^2}, 0, ..., 0)$$

so that

$$F(A\xi_1, A\eta_1) = F(\xi, \eta)$$

then, $F(\xi, \eta)$ will be a polynomial in the two variables t and $-\sqrt{1 - t^2}$, that is

$$F(\xi, \eta) = P(t, -\sqrt{1 - t^2})$$

which means that

$$P(t, -\sqrt{1 - t^2}) = P(t, \sqrt{1 - t^2})$$

Then P is a polynomial in t and $1 - t^2$, which is a polynomial in t so that

$$F(\xi, \eta) = \phi(t) = \phi((\xi, \eta))$$

5. Legendre polynomials

In this chapter, we will develop the relationship between spherical harmonics and Legendre polynomials as well as the kernel $F(\xi, \eta)$ of the orthogonal projector onto $\mathscr{H}_n$ can be represented in terms of nth degree Legendre polynomial.

Theorem 5.1. *Consider the homogeneous and harmonic polynomial $L_n(X)$ characterized by the following properties.*

(1) *Without loss of generality we let $\eta = (1, 0, ..., 0)$. Then $L_n(\eta) = 1$*
(2) *Let A' be any rotation leaving η fixed, that is $A'\eta = \eta$. Then $L_n(A'X) = L_n(X)$.*
The corresponding spherical harmonic $L_n(\xi)$ is uniquely defined by these properties and is a polynomial in $t = (\xi, \eta)$.

Proof. For any $\xi \in S^{d-1}$, let $r = \sqrt{x_1^2 + x_2^2 + ... + x_d^2}$, and ξ' be a unit vector orthogonal to η that is $(\xi', \eta) = 0$. We can express ξ in the form

$$\xi = t\eta + \sqrt{1-t^2}\xi'$$

for suitable t and ξ'. Note that $t = (\xi, \eta)$. We now expand $L_n(X)$ in the form

$$L_n(X) = \sum_{j=0}^{n} x_d^j A_{n-j}(x_1, x_2, ..., x_{d-1})$$

Let A' be a rotation matrix as described in property (2) in the statement of the theorem. Then,A' is a linear mapping:

$$A' : x_1, x_2 ..., x_d \to x_1, x_2^{'}, ..., x_d^{'}$$

In view of the fact that the transformation A' leaves $L_n(X)$ invariant, we get

$$0 = L_n(x) - L_n(Rx) = \sum_{j=0}^{n} x_d^j A_{n-j}(x_1, x_2, ..., x_{d-1})$$

By the linear independence of $\{x_d^j\}_{j=0}^d$, it must leave all $A_{n-j}(x_1, ..., x_{d-1})$ invariant. In other words, the A_{n-j} are invariant under all such rotations and it follows that A_{n-j} depends only on the radius r, that is,

$$A_{n-j}(x_1, ..., x_{d-1}) = C_{n-j}(x_1^2 + x_2^2 + ... + x_{d-1}^2)^{\frac{n-j}{2}}$$

so that $C_{n-j} = 0$ for all odd $n - j$. Otherwise these would not be polynomials. Hence either $A_n = 0$ if n is odd or $A_{n-1} = 0$ if n is even. We had seen earlier that the specifications of A_n and A_{n-1} uniquely determines $L_n(X)$ in theorem 4.5. In this case the specification of a single constant, either C_n or C_{n-1} depending on the parity of n, determines $L_n(X)$. We have now

$$L_n(\xi) = \sum_{j=0}^{n}{}' t^j (1-t^2)^{\frac{n-j}{2}} C_{n-j}$$

where $\sum'$ denotes that we sum only over such values of j for which $n - j$ is even. By property 1 with $t = 1$

$$L_n(\eta) = C_0 = 1$$

Thus, the single remaining constant is specified, and $L_n(X)$ uniquely determined as in theorem 4.5. Hence, the corresponding $L_n(\xi)$ is uniquely determined.

Remark 5.2. *The polynomial defined in the preceding theorem will be defined as the Legendre polynomial of degree n in d dimensions. We also could get the following properties of Legendre polynomials.*

(1) $1 = L_n(\eta) = P_n((\eta, \eta)) = P_n(1)$
(2) *From the homogeneity of Legendre polynomial and (1), we get* $P_n(-1) = (-1)^n P_n(1) = (-1)^n$

Now we are equipped to prove the Addition Theorem for Legendre Polynomials which basically says that every Legendre polynomial can be represented in terms of spherical harmonics.

Theorem 5.3. (Addition Theorem for Legendre Polynomial)
Let $\{S_{n,j}\}_{j=1}^{N(d,n)}$ be the orthonormal set of spherical harmonics of degree n. Then the Legendre polynomial of degree n, P_n can be written as

$$P_n((\xi, \eta)) = \frac{\omega_{d-1}}{N(d,n)} \sum_{j=1}^{N(d,n)} S_{n,j}(\xi) S_{n,j}(\eta). \tag{30}$$

Proof. We now need the function $F(\xi, \eta)$ defined in (26). Consider the normalized function $\frac{F(\xi,\eta)}{F(\eta,\eta)}$, we could check that this function satisfies the properties in theorem5.1: If A' is any rotation leaving η fixed we have

$$F(A'\xi, A'\eta) = F(A'\xi, \eta) = F(\xi, \eta)$$

Hence, by the theorem5.1, the normalized function $\frac{F(\xi,\eta)}{F(\eta,\eta)}$ is a uniquely determined Legendre polynomial denote P_n, i.e.

$$F(\xi, \eta) = F(\eta, \eta) P_n((\xi, \eta)) \tag{31}$$

To evaluate $F(\eta,\eta)$, we integrate $F(\eta,\eta)$ over the surface of the unit sphere in d dimensions.

$$\begin{aligned}\int_{|\eta|=1} F(\eta,\eta)d\omega_{d-1} &= \int_{|\eta|=1}\sum_{j=1}^{N(d,n)} S_{n,j}^2(\eta)d\omega_d \\ &= \sum_{j=1}^{N(d,n)}\int_{|\eta|=1} S_{n,j}^2(\eta)d\omega_{d-1} \\ &= N(d,n)\end{aligned}$$

The last equality holds since $\{S_{n,j}\}$ is orthonormal set. Then, we get

$$F(\eta,\eta) = \frac{N(d,n)}{\omega_{d-1}}$$

Hence, we get

$$P_n((\xi,\eta)) = \frac{\omega_{d-1}}{N(d,n)}F(\xi,\eta)$$

The result follows.

Remark 5.4. *From the equation (31), we see that the kernel $F(\xi,\eta)$ of the orthogonal projector onto $\mathscr{H}_n$ can be represented in terms of nth degree Legendre polynomial.*

In the above theorem, we showed that a given Legendre polynomial with degree n can be expressed in terms of spherical harmonics. In the following, we will show that every spherical harmonic can be represented in terms of Legendre polynomial.

Lemma 5.5. *Let $\{S_{n,j(\eta)}\}$ be a set of $k \le N(d,n)$ linearly independent spherical harmonics. Then there exists a set of k unit vectors $\{\eta_i\}$ such that the following $k \times k$ determinant does not vanish.*

$$|S_{n,j}(\eta_i)| \neq 0$$

Proof. We can certainly find η_1 so that $S_{n,1}(\eta_1) \neq 0$. Then we consider the spherical harmonic defined by

$$\begin{vmatrix} S_{n,1}(\eta_1) & S_{n,1}(\xi) \\ S_{n,2}(\eta_1) & S_{n,2}(\xi)\end{vmatrix}.$$

Since $S_{n,1}$ and $S_{n,2}$ are linearly independent, the above does not vanish identically and we select η_2 so that it does not vanish. In a similar fashion we can select η_3, η_4,...,η_k, by going on to higher determinants

$$\begin{vmatrix} S_{n,1}(\eta_1) & S_{n,2}(\xi) & ... & S_{n,n}(\xi) \\ S_{n,2}(\eta_1) & S_{n,2}(\xi) & ... & S_{n,n}(\xi) \\ \cdots & & & \\ S_{n,n}(\eta_1) & S_{n,2}(\xi) & ... & S_{n,n}(\xi)\end{vmatrix}$$

Then, the result follows.

By means of the above lemma we can prove the following theorem. The key point of the theorem is that every spherical harmonic can be represented in terms of the basic Legendre polynomials.

Theorem 5.6. *Every spherical harmonic $S_n(\xi)$ can be represented in the form*

$$S_n(\xi) = \sum_{k=1}^{N(d,n)} A_k P_n((\xi,\eta_k))$$

where $P_n(t)$ is the Legendre polynomial of degree n in d dimensions and the η_k are a set of suitable unit vectors.

Proof. According to the previous lemma we can select $\{\eta_i\}$ so that $|S_{n,j}(\eta_i)| \neq 0$ Then the linear equation system

$$P_n((\xi,\eta_k)) = \frac{\omega_{d-1}}{N(d,n)}\sum_{j=1}^{N(d,n)} S_{n,j}(\xi)S_{n,j}(\eta_k)\ k = 1,2,...,N(d,n) \tag{32}$$

is invertible and

$$S_{n,j}(\xi) = \sum_{k=1}^{N(d,n)} A_k P_n((\xi,\eta_k)),\ j = 1,2,...,N(d,n).$$

But every spherical harmonic of degree n can be expressed as a linear combination of $S_{n,j},\ j = 1,2,...,N(d,n)$. Hence,

$$S_n(\xi) = \sum_{k=1}^{N(d,n)} b_j S_{n,j}(\xi)$$

where $b_j = \int_{|\xi|=1} S_n(\xi)S_{n,j}(\xi)d\omega_d$.

Proposition 5.7. *For every spherical harmonic $S_n(\xi)$ of degree n, we have*

$$S_n(\xi) = \frac{N(d,n)}{\omega_d}\int_{|\eta|=1} S_n(\eta)P_n((\xi,\eta))d\omega_d \tag{33}$$

Proof. Let

$$S_n(\xi) = \sum_{k=1}^{N(d,n)} b_j S_{n,j}(\xi)$$

Using (32), the right hand side of (33) can be written as

$$\sum_{j=1}^{N(d,n)} S_{n,j}(\xi)\sum_{i=1}^{N(d,n)} b_i \int_{|\eta|=1} S_{n,j}(\eta)S_{n,i}(\eta)d\omega_d = \sum_{k=1}^{N(d,n)} b_j S_{n,j}(\xi) = S_n(\xi).$$

Up to now, we know that the spherical harmonics and Legendre polynomials could be expressed by each other.

6. Applications to Boundary Value Problems

In this section, we will first prove that all spherical harmonics $\{S_{n,j}(\xi)\}$ where $j = 1, 2, ..., N(d,n)$, and $n = 0, 1, 2, ...$ form a complete basis for the Hilbert space $L^2(S^{d-1})$. Then, We can develop certain expansion theorems in connection with the orthonormal set. In the end, we will see an application of spherical harmonics to the Dirichlet boundary condition problems of Laplace equation on a ball.

Definition 6.1. *we let $\mathscr{H}_n$ denote the vector space of all spherical harmonics of degree n in d dimension. i.e. $\mathscr{H}_n$ is the collection of all homogeneous harmonic polynomials of degree n restricted to the sphere S^{d-1}.*

Definition 6.2. $L^2(S^{d-1}) := \{f(\xi) : S^{d-1} \to R$ *such that* $\int_{S^{d-1}} |f(\xi)|^2 d\omega_d < \infty\}$.

Theorem 6.3. $L^2(S^{d-1})$ *is a Hilbert space with inner product defined by* $(f,g) = \int_{|\xi|=1} f(\xi)g(\xi)d\omega_d$.

we omit the proof of this theorem.

In the following, we will show that $L^2(S^{d-1}) = \mathscr{H}_0 \oplus \mathscr{H}_1 \oplus \mathscr{H}_2 \oplus ... \oplus \mathscr{H}_n \oplus ...$ which we based on to develop the expansion.

Theorem 6.4. *Let $f(\xi)$ be a real continuous function on $|\xi| = 1$. If $(f(\xi), S_{n,j}(\xi)) = 0$ for all spherical harmonics in $\{S_{n,j}(\xi)\}$ then $f(\xi)$ vanishes identically.*

Proof. Without loss of generality we can assume that there exists an η such that $f(\eta) > 0$. By the continuity of $f(\xi)$ there exists a neighborhood of ξ defined by $\tau \le (\xi,\eta) \le 1$ such that $f(\xi) \ge a > 0$ for all ξ in that neighborhood.
Consider the function

$$\phi(t) = 1 - \frac{(1-t)^2}{(1-\tau)^2} \quad \tau \le t \le 1$$

$$\phi(t) = 0, \ -1 \le t \le \tau$$

There exists then a positive constant b such that

$$\int_{|\xi|=1} f(\xi)\phi((\xi,\eta))d\omega_d \ge b > 0.$$

Using the Weierstrass approximation theorem we can construct a polynomial $p(t)$ so that

$$|\phi(t) - p(t)| \le \epsilon, \ -1 \le t \le 1.$$

It follows that if $|f(\xi)| \le M$, then

$$|\int_{|\xi|\le 1} f(\xi)[\phi((\xi,\eta)) - p((\xi,\eta))]d\omega_d| \le M\omega_d\epsilon$$

and

$$\int_{|\xi|=1} f(\xi)p((\xi,\eta))d\omega_d \ge b - M\omega_d\epsilon > 0$$

for sufficiently small ϵ. But using the completeness properties of the Legendre polynomials we have

$$p(t) = \sum_{k=0}^{n} A_k P_k(t)$$

so that

$$\int_{|\xi|=1} f(\xi) \sum_{k=0}^{n} A_k P_k((\xi,\eta)) d\omega_d = 0$$

by hypothesis. But (1) and (2) contradict one another. Hence $f(\xi) = 0$.

From the definition 4.3, we know $\{S_{n,j}(\xi)\}_{j=1}^{N(d,n)}{}_{n=0}^{\infty}$ is a closed set, and by the theorem 2.4, we get the conclusion that $\{S_{n,j}(\xi)\}_{j=1}^{N(d,n)}{}_{n=0}^{\infty}$ forms a closed orthonormal set. In the following, I will briefly state the idea to extend the conclusion in 6.4 to $L^2(S^{d-1})$, then, we could say that $\{S_{n,j}(\xi)\}_{j=1}^{N(d,n)}{}_{n=0}^{\infty}$ forms a complete orthonormal basis of Hilbert space $L^2(S^{d-1})$ or we could say that $L^2(S^{d-1}) = \mathscr{H}_0 \oplus \mathscr{H}_1 \oplus \mathscr{H}_2 \oplus ... \oplus \mathscr{H}_n \oplus ....$

Let $f(\xi) \in L^2(S^{d-1})$, $\{S_{n,j}(\xi)\}_{j=1}^{N(d,n)}{}_{n=0}^{\infty}$ be the complete orthonormal set of $L^2(S^{d-1})$, then, we have the following expansion

$$f(\xi) = \sum_{n=0}^{\infty} \sum_{j=1}^{N(d,n)} C_{n,j} S_{n,j}(\xi)$$

with certain coefficients $C_{n,j}$ that we will derive later. Now extending the above theorem to all functions $f(\xi) \in L^2(S^{d-1})$.

First, let me state a certain fact about square-integrable functions without proof. For every ϵ one can find a continuous function $g(\xi)$ so that

$$\int_{|\xi|=1} |f(\xi) - g(\xi)|^2 d\omega_d \leq \epsilon.$$

With this and the theorem 6.4 one can show that

$$\lim_{K\to\infty} \int_{|\xi|=1} |f(\xi) - \sum_{n=0}^{K} \sum_{j=1}^{N(d,n)} C_{n,j} S_{n,j}(\xi)|^2 d\omega_d = 0$$

then the same conclusion for all such functions $f(\xi) \in L^2(S^{d-1})$ follows.

Now, let's To obtain the coefficient $C_{n,j}$,

$$\begin{aligned}\int_{|\xi|=1} f(\xi) S_{n,j}(\xi) d\omega_d &= \sum_{m=0}^{\infty} \sum_{k}^{N(d,m)} \int_{|\xi|=1} C_{m,k} S_{m,k}(\xi) S_{n,j}(\xi) d\omega_d \\ &= \sum_{m=0}^{\infty} \sum_{k}^{N(d,m)} C_{m,k} \int_{|\xi|=1} S_{m,k}(\xi) S_{n,j}(\xi) d\omega_d \\ &= C_{n,j}\end{aligned}$$

since $\{S_{n,j}(\xi)\}_{j=1}^{N(d,n)}{}_{n=0}^{\infty}$ is orthormonal.

Using the conclusion we get above, now we can develop expansions. by the addition theorem of Legendre polynomial 5.3, we could write the expansion of f as:

$$\begin{aligned}f(\xi) &= \sum_{n=0}^{\infty} \sum_{j=1}^{N(d,n)} C_{n,j} S_{n,j}(\xi) \\ &= \sum_{n=0}^{\infty} \sum_{j=1}^{N(d,n)} \left(\int_{|\eta|=1} f(\eta) S_{n,j}(\eta) d\omega_d \right) S_{n,j}(\xi) \\ &= \sum_{n=0}^{\infty} \sum_{j=1}^{N(d,n)} \left(\int_{|\eta|=1} f(\eta) S_{n,j}(\eta) S_{n,j}(\xi) d\omega_d \right) \\ &= \sum_{n=0}^{\infty} \int_{|\eta|=1} f(\eta) \left(\sum_{j=1}^{N(d,n)} S_{n,j}(\eta) S_{n,j}(\xi) \right) d\omega_d \\ &= \sum_{n=0}^{\infty} \int_{|\eta|=1} f(\eta) P_n((\xi,\eta)) d\omega_d \\ &= \sum_{n=0}^{\infty} \int_{|\eta|=1} f(\eta) F(\xi,\eta) d\omega_d\end{aligned}$$

Now, we see that when we compute the expansion of f, we only need the kernel $F(\xi,\eta)$ instead of the explicit expression of basis. Hence, we see that the function $F(\xi,\eta)$ defined in (26) has similar properties and plays a

similar role as does $K_n(x, y)$, defined by the Christoffel-Darboux formula in the theory of orthogonal polynomials. Now let's see the application on boundary value problems of Laplace equation on a ball.
We seek a harmonic function V satisfying a boundary condition on the unit sphere. That is

$$\Delta_d V = 0 \tag{34}$$

with the boundary condition $V = f(\xi)$ on $\{|r| = 1\}$.
Using the above conclusion, we see that

$$V = \sum_{n,j} r^n C_{n,j} S_{n,j}(\xi) \tag{35}$$

must satisfy the boundary condition and by construction must also be harmonic. Hence (35) is the solution of (34).

References

[1] Hochstadt, Harry. The functions of mathematical physics. Courier Corporation, 2012.
[2] Müller, Claus. Spherical harmonics. Vol. 17. Springer, 2006.
[3] Arfken, George B., and Hans J. Weber. "Mathematical methods for physicists." (1999): 165-169.
[4] Hofmann-Wellenhof, Bernhard, and Helmut Moritz. Physical geodesy. Springer Science Business Media, 2006.
[5] Structural forces in liquid crystalline blue phases

Approximation Theory, Department of Mathematics, University of South Carolina
Email address: chunyan@email.sc.edu