Weak solution via Galerkin Method

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1 Methods to show the existence and uniqueness for a pde equation

a. For elliptic equations, one could apply Lax-Milgram Lemma to show the existence and uniqueness of the weak solution of the corresponding weak formulation.

b. For linear parabolic equations and linear hyperbolic equations, one can show the existence and uniqueness of weak solution of the corresponding weak formulation via Galerkin approximation and energy estimation. For nonlinear case, we need to pay more effort to get the same conclusion.

c. For nonlinear pde, there is a general method to prove the existence and uniqueness which is Semigroup theory and Banach fixed point theorem. One can use this method to show the existence and uniqueness of mild solution. (Hence, the questions follows, what's a mild solution? Why it is called a mild solution? Where it comes from? what kind of physical meaning does it have?)

1.1 Detail for case a.

- Find weak formulation of the given equation. Key point: Boundary condition + divergence theorem + specify norm is important.
- Verify the conditions in Lax-Milgram Lemma

1.2 Detail for case b.

We first state the procedure for linear pde, the nonlinear case will be developed later.

- Find weak formulation of the given equation
  Key point: u_t denotes weak time derivative w.r.t time variable t + solution space + test space + divergence theorem which will need boundary conditions.
- Apply Galerkin approximation to construct the finite-D V_m approximated problem/equation of the original equation to get a IVP of ODE. Then, one can get the existence and uniqueness of solution u_m for this ode via Cauchy Lipschtz theorem of ODE theory.
  Key point: Choose V_m. If we choose V_m such that basis v_i has L^2 orthogonality and H^1 orthogonality, then, the coresponding ODE system have very nice structure which is diagonal matrix for coeff/spectral method. If we pick V_m be polynomial space, then, we get finite element method.

  Remark: Later, we will talk about exactly what's spectral method and Finite element method and why are they so called the names?
- Show the existence of original weak formulation

1. Find the possible solution via Energy estimate + Banach Alaoglu Theorem + the definition of weak convergence and weak * convergence to pass the limit of the equation, i.e. $(u_m, u_{m_t}) \to (u, u')$ weakly as $m \to \infty$

   a. uniform boundedness of u_m
   Goal: get uniform boundedness of u_m in $L^2H^1 norm$ and $L^\infty H^1 norm$
   Method: Energy estimate + Gronwall's inequality (In this case, state clearly norm is very important, $L^2 norm$ or $H^1 norm$)

   b. uniform boundedness of u_{m_t} in L^2H^{-1} norm
   Definition $||u_{m_t}||_{H^{-1}} = sup_{v\in H^1, v\neq 0} \dfrac{|< u_{m_t}, v >|}{||v||_{H^1}}$
   Key point: H^{-1} norm/norm of a functional is defined for all $v \in H^1$, but we only have equality for $v \in V_m$ from the approximated Finite-D weak formulation , hence, we need direct sum decomposition of Hilbert space H^1.
   Let $H^1 = V_m + V_m^{'}$, then, for all $v \in H^1$,

   $$< u_{m_t}, v >=< u_{m_t}, v_1 > + < u_{m_t}, v_2 >=< u_{m_t}, v_1 >$$

   The last equality is due to the L^2 orthogonality. Finally, One can show the uniform boundedness via Cauchy inequality and Holder's inequality.

2. check the possible solution is the solution which means the limit point (u, u') satisfies the equation. We not only need to pass to the limit to the equation, show that V_m will approach to H^1 as $m \to \infty$ which is true if your solution space is a separable Hilbret space and choose $V_m = span\{v_1, ..., v_m\}$. One can show this in a rigorous way via the following procure.
   Define $A =< u_t, v > +B[u(t,\cdot), v] =< f, v > \forall v \in H_0^1(\Omega)$
   Define $B_m =< u_{m_t}, v > +B[u_m(t,\cdot), v] =< f, v > \forall v \in H_0^1(\Omega)$
   Define $C_m =< u_{m_t}, v > +B[u_m(t,\cdot), v] =< f, v > \forall v \in V_m$
   Show $|A - B_m| \to 0 as m \to \infty$ and $|B_m - C_m| \to 0 as m \to \infty$ which will need $lim||v - v_m||_{H^1} = 0$ follows with the fact that V_m is the orthogonal space of H^1

- Show the uniqueness
  One can use energy estimate to show the uniqueness of u_m for all m, then pass to the limit to get the uniqueness of solution u.

  - $E_m(t) \leq 0$ by energy estimate, hence, $u_1^m - u_2^m \equiv 0 \ \forall m$
  - pass to the limit, for all test function v,

  $$0 = \int_0^T \int_\Omega (u_1^m - u_2^m) v dx dt \to \int_0^T \int_\Omega (u_1 - u_2) v dx dt$$

  as $m \to \infty$, hence the uniqueness follows.

Remark Why we first show the uniqueness for u_m for all m when we show the uniqueness of solution?
Since, when we do energy estimate, we need certain smoothness of u to allow us do integration by parts. The solution u of pde may not satisfy but u_m does.

For nonlinear pde which has nonlinear term $b(u, v, w)$,There are two trouble stop us to get the existence. We need new powerful tools to kill the trouble.

- uniform boundedness of $||u_{m_t}||_{L^2H^{-1}}$ is hard to get, so we first could use Sobolve embedding $H^1 \subset L^4$ to get the uniform boundedness for $||u_{m_t}||_{L^1H^{-1}}$. We could use other tools to get uniform boundedness of $||u_{m_t}||_{L^2H^{-1}}$ which was not covered in the lecture notes. The conclusion follows from this uniform boundedness is $u_{m_t} \to u'$ weakly * converges in MH^{-1} (We need to explain what's this weakly * in MH^{-1} means)

- $b(u_m, v, w) \to b(u, v, w)$ in order to passing to the limit, we need something. In the following case, we need strong convergence which comes from Aubin-Lions Lemma.

For nonlinear pde, we need new tools combined with energy estimate to get the uniqueness. We use a concrete example to show how to get the uniqueness for nonlinear pde.

$$Y' + friend \leq evil$$

Key point: evil could be absorbed by friends or bounded by Y where 'friend' is Laplace operator.

- sobolev embedding
- mean value theorem $f(v) - f(u) = f'(\theta)(v - u)$

Remark In Galerkin Approximation, choose proper V_m is the key point.

I will write another note for all the detail of this whole part with concrete examples.

1.3 Detail for case c.

Definition

- Semi-Group The collection of all $S(t)$ which has property $S(s)S(t) = S(t+s) = S(t)S(s)$ for $s, t > 0$.

- Mild solution
  For equation: $u_t - \gamma u_{xx} = -uu_x$, with IC $u(t=0, x) = g(x)$ the solution satisfies:

  $$u(t,x) = S(t)g + \int_0^t [-S(t-s)u(s)u_x(s)]ds$$

  is called mild solution.

- Banach Fixed Point Theorem
  Let X be a Banach space, $T : X \to X$ is a linear operator. Suppose exist $q \in [0, 1)$ such that $\forall x, y \in X$, $||Tx - Ty||_X \leq q||x - y||_X$, then, T has a unique fix point, i.e. $Tx = x$.

  Proof. Let $x_{n+1} = Tx_n$, with $x_0 \in X$. We have $||x_{n+1} - x_n||_X = ||Tx_n - Tx_{n-1}||_X \leq q||x_n - x_{n-1}||_X ... \leq q^k||x_1 - x_0||_X \to 0$ as $k \to \infty$. Hence, $\{x_n\}$ i.e. $\{Tx_n\}$ is a Cauchy sequence in X. Then, there exist a unique limit point in $\bar{x} \in X$ by the completeness of X.

  $$x = lim x_{n+1} = lim Tx_n = Tx$$

  □

  Remark This is a very strong argument which gives us existence and uniqueness.

Now, the question follows:

- How do we pick proper Banach space X?

- How do we choose the initial value $x_0 \in X$ to make contraction mapping assumption to be true and to make the iteration method converges quickly?